OPTION PRICING WITH NON-GAUSSIAN DISTRIBUTION-NUMERICAL APPROACH

Preliminary Exam Report

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September 11, 2012
Abstract

This report is intended to solve the option pricing problem based on new models of underlying asset prices rather than traditional Gaussian-based model.

Empirical evidences have proved that the distribution of the log return of asset prices has three obvious properties which are contradictory to Gaussian distribution: heavy tail, skewness and volatility clustering. To better simulate the distribution of log return, we introduce α-stable distribution and other tempered stable distributions which are more flexible, heavy tailed and skewed.

Based on the infinitely divisible property of these stable distributions, various Lévy processes can be generated as the basis for simulating the price path of underlying asset instead of Brownian motion. Then, Escher theorem is introduced to derive equivalent martingale measure. Exponential Lévy models are obtained to simulate stock price and corresponding martingale measure is available.

The option pricing model based on risk neutral formula is an efficient method which can also be applied to exponential Lévy models. Based on the equivalent martingale measure derived from Escher theorem, the analytic expression of option price as well as the integro-differential equation can be obtained.

Besides on all these analysis, there are still a lot of area deserve further research. First, to preserve the positive qualities of α-stable distribution and abandon the drawbacks, we should explore truncated stable distribution and other generalized tempered stable distributions. Second, ARMA and GARCH models should be introduced for option pricing to cover the volatility clustering property. Third, apply these stable models to high frequency data and develop efficient algorithms to solve corresponding large scale problem.
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1 Introduction

Cumulative records have led to doubt about traditional financial models based on Gaussian distribution. In this section, we will explore both former research results and the motivation to introduce stable distribution models and Lévy process.

1.1 Motivation

Louis Bachelier (1900) contributes to the construction of the random-walk model for security and commodity markets and formed the basis of diffusion process. Since then, there is a tradition to assume that prices in speculative markets, such as grain and securities markets, behave very much like random walks. Based on relative properties of random walk and central limit law, many researchers believe that price changes across differencing intervals such as a day, a week, or a month will be normally distributed. However, Benoît Mandelbrot (1963) revealed some contradictory empirical observations: firstly, the empirical distributions of price changes are usually too “peaked” to be relative to samples from Gaussian populations; secondly, the tails of the distributions of price changes are in fact so extraordinarily long that the sample second moments typically vary in an erratic fashion. And he suggests a better approximation stable Pareto distribution. Eugene F. Fama (1963) further proves the stable Pareto hypothesis theoretically and empirically.

In sum, despite the popularity of normal model, it cannot explain three observed facts regarding asset returns: fat-tailed, skewed and volatility clustering, which can partly be covered by stable distribution[11].

The tails of the distributions are where the extreme values occur and they are more likely to happen than these predicted by Gaussian distributions empirically. Accordingly, fat tails can help explain larger price fluctuations for stocks over short time periods than in fundamental economic analysis.

The Gaussian distribution is a symmetric distribution which means the left side of the distribution is the mirror image of the right side. Typically in a skewed distribution, one tail of the distribution is much longer than the other tail of the probability distribution. And this longer tail is referred to as fat tails and has greater probability of extreme values occurring.

Volatility clustering behavior refers to the tendency of large changes in asset prices to be followed by large changes and small changes to be followed by small changes. Observations of this type in financial time series have led to the use of
GARCH models in financial forecasting and derivatives pricing.

From Figure 1.1, we can find out that high frequency market data of Standard & Poor’s 500 index is too peaked to be fitted by Gaussian distribution. Moreover, there are more points locate on the tails than predicted by Gaussian distribution.

1.2 Organization

In this report, we mainly focused on the application of non-Gaussian distribution for modeling the behavior of stock price returns and obtain the corresponding option pricing result.

Starting from the motivation of introducing non-Gaussian distribution, we reveal three contradictory empirical observations which are supported by data of Standard & Poor’s 500 index.

In Section 2, we go into depth regarding the definitions and properties of stable distribution and tempered stable distribution. Then, we discuss how to simulate $\alpha$-stable random variable and the estimation of stable parameters according to the market data.

In Section 3, we investigate the definition of Lévy process and the generation
of Lévy process based on infinitely divisible distributions. Furthermore, Esscher theorem is introduced to derive equivalent martingale measure which set basis for developing exponential Lévy model.

In Section 4, we introduce the no-arbitrage pricing formula for European option and obtain numerical option pricing methods based on exponential Lévy model.

The last section describes three major future research aspects: first, introduce new tempered stable distribution; second, introduce ARMA and GARCH models; third, improve the efficiency of current option pricing algorithms.
2 Stable and Tempered Stable Distribution

The normal distribution has been widely applied in modeling the return distribution of assets. However, it is not consistent with the observed behavior and cannot describe the skewed and fat-tailed properties of the empirical distribution. The $\alpha$-stable distribution can be used as an alternative to cover the drawbacks. In order to obtain a well-defined model for pricing options, the mean and variance should exist. Thus, the tempered stable distributions are introduced.

2.1 $\alpha$-Stable Distribution

2.1.1 Definition of $\alpha$-Stable Random Variable

Given independent and identically distributed random variables $X_1, X_2, \cdots, X_n$ and $X$, then random variable $X$ is said to follow an $\alpha$-stable distribution if there exist a positive constant $C_n$ and a real number $D_n$ such that the following relation holds:

$$X_1 + X_2 + \cdots + X_n \overset{d}{=} C_nX + D_n \quad (2.1)$$

where $\overset{d}{=}$ denotes equality in distribution, and constant $C_n = n^{\frac{1}{\alpha}}$ determines the stability property. When $\alpha = 2$, it is the Gaussian case; when $0 < \alpha < 2$, we have the non-Gaussian case.

Generally, $\alpha$-Stable distribution does not have closed form density function and is expressed by characteristic function:

$$\phi_{\text{stable}}(u; \alpha, \sigma, \beta, \mu) = E[e^{iuX}]$$

$$= \begin{cases} 
\exp (i\mu u - |\sigma u|^\alpha (1 - i\beta \text{sign}(u) \tan \frac{\alpha \pi}{2})) & \alpha \neq 1 \\
\exp (i\mu u - |u| (1 + i\beta^2 \frac{2}{\pi} \text{sign}(u) \ln |u|)) & \alpha = 1 
\end{cases} \quad (2.2)$$

where

$$\text{sign} t = \begin{cases} 
1, & t > 0 \\
0, & t = 0 \\
-1, & t < 0 
\end{cases} \quad (2.3)$$

The distribution is characterized by four parameters:
• $\alpha$: the index of stability or the shape parameter, $\alpha \in (0, 2)$

• $\beta$: the skewness parameter, $\beta \in [-1, 1]$

• $\sigma$: the scale parameter, $\sigma \in (0, +\infty)$

• $\mu$: the location parameter, $\mu \in (-\infty, +\infty)$

Because of the four parameters, the $\alpha$-stable distribution is highly flexible and suitable for modeling non-symmetric, highly kurtotic and heavy-tailed data.

### 2.1.2 Properties of $\alpha$-Stable Random Variable

$\alpha$-stable distribution has some unique properties which worth discussion.

• (Power Tail Decay) The tail of the density function of $\alpha$-stable distribution decays like a power function which is slower than the exponential decay

\[
P(|X| > x) \propto Cx^{-\alpha}, \quad x \to \infty.
\]

(2.4)

• (Raw Moments) The raw moments satisfy the property

\[
E|X|^p < \infty, \quad 0 < p < \alpha
\]

\[
E|X|^p = \infty, \quad p \geq \alpha
\]

(2.5)

• (Stability) The distribution is preserved under linear transformation. Note the standard Central Limit Theorem does not apply to non-Gaussian stable distribution, i.e. the sum of non-Gaussian stable random variables is still a stable random variable. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed with $X_i \sim S_\alpha (\sigma_i, \beta_i, \mu_i)$, then:

- The distribution of $Y = \sum_{i=1}^n X_i$ is $\alpha$-stable with the index of stability $\alpha$ and parameters

\[
\beta = \frac{\sum_{i=1}^n \beta_i \sigma_i^\alpha}{\sum_{i=1}^n \sigma_i^\alpha}, \quad \sigma = \left(\sum_{i=1}^n \sigma_i^\alpha\right)^{\frac{1}{\alpha}}, \quad \mu = \sum_{i=1}^n \mu_i
\]

(2.6)
The distribution of \( Y = X_1 + a \) for some constant \( a \) is \( \alpha \)-stable with the index of stability \( \alpha \) and parameters

\[
\beta = \beta_1, \quad \sigma = \sigma_1, \quad \mu = \mu_1 + a
\]  

(2.7)

The distribution of \( Y = aX_1 \) for some constant \( a \) is \( \alpha \)-stable with the index of stability \( \alpha \) and parameters

\[
\beta = (\text{sign} a) \beta_1, \quad \sigma = |a| \sigma_1, \quad \mu = \begin{cases} 
  a \mu_1, & \alpha \neq 1 \\
  a \mu_1 - \frac{2}{\pi} a (\ln |a|) \sigma_1 \beta_1, & \alpha = 1
\end{cases}
\]

(2.8)

The distribution of \( Y = -X_1 \) for some constant \( a \) is \( \alpha \)-stable with the index of stability \( \alpha \) and parameters

\[
\beta = -\beta_1, \quad \sigma = \sigma_1, \quad \mu = \mu_1
\]  

(2.9)

- A generalized theorem is that two admissible parameter quadruples \((\alpha, \beta, \sigma, \mu)\) and \((\alpha, \beta, \sigma', \mu')\) uniquely determine real numbers \( a > 0 \) and \( b \) such that

\[
X (\alpha, \beta, \sigma, \mu) \overset{d}{=} aX (\alpha, \beta, \sigma', \mu') + b
\]  

(2.10)

where

\[
a = \frac{\sigma}{\sigma'}, \quad b = \begin{cases} 
  \mu - \mu' \frac{\sigma}{\sigma'}, & \alpha \neq 1 \\
  \mu - \mu' \frac{\sigma}{\sigma'} + \sigma \beta^2 \frac{\sigma}{\sigma'} \ln \frac{\sigma}{\sigma'}, & \alpha = 1
\end{cases}
\]

(2.11)

2.2 Simulation of \( \alpha \)-Stable Random Variable

The method for simulating \( \alpha \)-stable random variable is introduced in many literatures. Here, we take advantage of one of the most efficient algorithms given by Weron[13, 14].

Based on the theorem that given \( \gamma_0 = -\frac{\pi}{2} \beta_2 \frac{K(\alpha)}{\alpha} \), \( \gamma \) be uniformly distributed on \((-\frac{\pi}{2}, \frac{\pi}{2})\) and \( W \) be an independent exponential random variable with mean
1. Then

\[ X = \sin \alpha (\gamma - \gamma_0) \left( \frac{\cos (\gamma - \alpha (\gamma - \gamma_0))}{W} \right)^{\frac{1-\alpha}{\alpha}} \] (2.12)

is \( S_{\alpha}(1, \beta_2, 0) \) for \( \alpha \neq 1 \).

\[ X = \left( \frac{\pi}{2} + \beta_2 \gamma \right) \tan \gamma - \beta_2 \log \left( \frac{\pi}{2} + \beta_2 \gamma \right) \] (2.13)

is \( S_1(1, \beta_2, 0) \).

Thus, the algorithm for generating stable random variable can be summarized as:

- Generate a random variable \( U \) uniformly distributed on \((-\pi/2, \pi/2)\) and an independent exponential random variable \( E \) with mean 1.
- For \( \alpha \neq 1 \), compute
  \[ X = S_{\alpha, \beta} \sin \left( \alpha (U + B_{\alpha, \beta}) \right) \left( \frac{\cos (U - \alpha (U + B_{\alpha, \beta}))}{E} \right)^{\frac{1-\alpha}{\alpha}}, \] (2.14)
  where \( B_{\alpha, \beta} = \frac{\arctan(\beta \tan \frac{\pi U}{\alpha})}{\alpha} \), and \( S_{\alpha, \beta} = \left( 1 + \beta^2 \tan^2 \frac{\pi \mu}{2} \right)^{\frac{1}{2\alpha}} \).
- For \( \alpha = 1 \), compute
  \[ X = \frac{2}{\pi} \left[ \left( \frac{\pi}{2} + \beta U \right) \tan U - \beta \log \left( \frac{\pi}{2} + \beta U \right) \right]. \] (2.15)

Generalize scale and location

\[ Y = \begin{cases} \sigma X + \mu, & \alpha \neq 1 \\ \sigma X + \frac{2}{\pi} \beta \sigma \log \sigma + \mu, & \alpha = 1. \end{cases} \] (2.16)

According to this algorithm, we have implemented a very efficient program to generate \( \alpha \)-stable random variables with various parameter groups and come up with the distribution function.
Figure 2.1: Effect of $\alpha$ and $\beta$ on the Distribution

Figure 2.2: Effect of $\sigma$ and $\mu$ on the Distribution
2.3 Estimation of Parameters

Given a sample of observations \( S = \{x_1, x_2, \ldots, x_N\} \) and assume the sample is independent and identically distributed under \( \alpha \)-stable distribution \( S_\alpha (\sigma, \beta, \mu) \). We need to provide estimation \( \hat{\alpha}, \hat{\sigma}, \hat{\beta}, \hat{\mu} \) of four stable parameters. There are two relative accurate and efficient methods: sample characteristic function methods and maximum likelihood method.

2.3.1 Sample Characteristic Function Method

This method based on the fact that characteristic function is an expectation. It is the expected value of the random variable \( e^{iuX} \) where \( X \) follows \( \alpha \)-stable distribution. Then define the sample characteristic function

\[
\hat{\phi}_X(u) = \frac{1}{N} \sum_{j=1}^{N} e^{iu x_j}. \tag{2.17}
\]

\( |\hat{\phi}_X(u)| \) is bounded by unity all moments of \( \hat{\phi}_X(u) \) are finite and, for any fixed \( u \), it is the sample average of independent and identically distributed random variables \( e^{iu x_j} \). Hence, by the law of large number, \( \hat{\phi}_X(u) \) is a consistent estimator of the characteristic function \( \phi_X(u) \).

From transformations of the characteristic function, there is a method called moment estimation. We have for all \( \alpha \)

\[
|\phi_X(u)| = \exp(-\sigma^\alpha |u|^\alpha). \tag{2.18}
\]

Thus,

\[
-\log |\phi_X(u)| = \sigma^\alpha |u|^\alpha. \tag{2.19}
\]

Assume \( \alpha \neq 1 \), choose two nonzero values of \( u \), such that \( u_1 \neq u_2 \) and for \( k = 1, 2 \)

\[
-\log |\phi_X(u_k)| = \sigma^\alpha |u_k|^\alpha. \tag{2.20}
\]

Solving these two equations for \( \hat{\alpha} \) and \( \hat{\sigma} \)

\[
\hat{\alpha} = \frac{\log \frac{\log |\hat{\phi}(u_1)|}{\log |\hat{\phi}(u_2)|}}{\log \frac{u_1}{u_2}} \tag{2.21}
\]
and
\[ \log \hat{\sigma} = \log |u_1| \log \left( - \log \left| \hat{\phi}(u_2) \right| \right) - \log |u_2| \log \left( - \log \left| \hat{\phi}(u_1) \right| \right) / \log \left| \frac{u_1}{u_2} \right|. \] (2.22)

The estimation of \( \hat{\beta} \) and \( \hat{\mu} \) come from the real part and imaginary part of \( \phi_X(u) \) with similar method, we have
\[ \text{Re} \left( \phi_X(u) \right) = \exp (-|\sigma \mu|^\alpha) \cos \left( \mu u + |\sigma u|\beta \text{sign}(u) \tan \frac{\pi \alpha}{2} \right) \] (2.23)
and
\[ \text{Im} \left( \phi_X(u) \right) = \exp (-|\sigma \mu|^\alpha) \sin \left( \mu u + |\sigma u|\beta \text{sign}(u) \tan \frac{\pi \alpha}{2} \right) \] (2.24)
then, we have
\[ \left( \frac{\arctan \left( \frac{\text{Im} \left( \phi_X(u) \right)}{\text{Re} \left( \phi_X(u) \right)} \right)}{\mu u + |\sigma u|\beta \text{sign}(u) \tan \frac{\pi \alpha}{2}} \right) = \mu u + |\sigma u|\beta \text{sign}(u) \tan \frac{\pi \alpha}{2}. \] (2.25)

In this moment estimation method, the value \( u_1 = 0.2, u_2 = 0.8, u_3 = 0.1 \) and \( u_4 = 0.4 \) are proposed in the simulation study[4].

Another method is based on regression[7] which is more accurate, from the characteristic function, we have
\[ \log \left( - \log |\phi_X(u)|^2 \right) = \log (2\sigma^\alpha) + \alpha \log |u| \] (2.26)
assume \( y_k = \log \left( - \log |\phi_X(u_k)|^2 \right) \) and \( w_k = \log |u_k| \), then
\[ y_k = m + \alpha w_k + \epsilon_k \] (2.27)
where \( \{u_k\} \) is an appropriate set of real numbers, \( m = \log (2\sigma^\alpha) \) and \( \epsilon_k \) denotes an error term. One possible set is \( u_k = \frac{\pi k}{25}, k = 1, 2, \cdots, K \).

Once \( \hat{\alpha} \) and \( \hat{\sigma} \) are obtained, estimation of \( \hat{\beta} \) and \( \hat{\mu} \) can be obtained with regression of
\[ \left( \frac{\arctan \left( \frac{\text{Im} \left( \phi_X(u) \right)}{\text{Re} \left( \phi_X(u) \right)} \right)}{\mu u + |\sigma u|\beta \text{sign}(u) \tan \frac{\pi \alpha}{2}} \right) = \mu u + |\sigma u|\beta \text{sign}(u) \tan \frac{\pi \alpha}{2}. \] (2.28)
2.3.2 Maximum Likelihood Method

For a sample of observations, the maximum likelihood estimate[2] of the parameter vector \( \theta = (\alpha, \sigma, \beta, \mu) \) is obtained by maximizing the log-likelihood function

\[
L_\theta = \sum_{j=1}^{N} \log \hat{f}_X(x_j; \theta)
\]

where \( \hat{f}_X(x_j; \theta) \) is the stable density function. The tilde denotes the fact that we do not know the explicit form of the possibility density function and can only be obtained numerically.

The following equation describes the relation between possibility density function and characteristic function via Fourier-inverse transformation

\[
f_X(x; \theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixu} \phi_X(u; \theta) \, du. \tag{2.30}
\]

Since \( \phi_X(-u) = \overline{\phi_X(u)} \), then

\[
f_X(x) = \frac{1}{\pi} \text{Re} \left( \int_{0}^{\infty} e^{-uix} \phi_X(u) \, du \right) \tag{2.31}
\]

By discrete numerical integration

\[
f_X(x) \approx \hat{f}_X(x) = 2\text{Re} \left( \sum_{j=0}^{N-1} e^{-2\pi i (\frac{jK}{N})} \phi_X \left( \frac{2\pi jK}{N} \right) \frac{K}{N} \right) \tag{2.32}
\]

where \( K \) is some large real number and \( N \) is a positive integer with \( N > K \).

Let \( x_k = \frac{k-N}{K}, \quad k = 0, 1, 2, \cdots, N - 1 \), then

\[
e^{-2\pi i x_k \left( \frac{K}{N} \right)} = e^{-2\pi i \left( \frac{k}{N} \right)} \tag{2.33}
\]

and

\[
\hat{f}_X(x_k) = 2\text{Re} \left( \sum_{j=0}^{N-1} e^{-2\pi i (\frac{j}{N})} \phi_X \left( \frac{2\pi jK}{N} \right) \frac{K}{N} \right) \tag{2.34}
\]

which can be calculated efficiently by Fast Fourier Transformation algorithm.

2.4 Tempered Stable Distribution

Given \( \alpha \in (0, 1) \cup (1, 2), C, \lambda_+, \lambda_- > 0 \) and \( m \in \mathbb{R} \), random variable \( X \)
follows classical tempered stable (CTS) distribution if the characteristic function is

$$
\phi_X (u) = \phi_{CTS} (u; \alpha, C, \lambda_+, \lambda_-, m)
$$

$$
= \exp (ium - iuC \Gamma (1 - \alpha) (\lambda_+^{\alpha - 1} - \lambda_-^{\alpha - 1}))
+ C \Gamma (-\alpha) ((\lambda_+ - iu)^\alpha - \lambda_+^\alpha + (\lambda_- + iu)^\alpha - \lambda_-^\alpha)) \tag{2.35}
$$

and denote $X \sim CTS (\alpha, C, \lambda_+, \lambda_-, m)$. Then the cumulants of $X$ are

$$
c_1 (X) = m
$$

$$
c_n (X) = C \Gamma (n - \alpha) (\lambda_+^{n-\alpha} + (-1)^n \lambda_-^{n-\alpha}), \quad n = 2, 3, \ldots \tag{2.36}
$$

Among all the parameters:

- $m$ is expectation which determines the location of the distribution.
- $C$ is the scale parameter.
- $\lambda_+, \lambda_-$ control the rate of decay on the positive and negative tails. If $\lambda_+ > \lambda_-$ or $\lambda_+ < \lambda_-$, then the distribution is skewed to the left or right correspondingly. If $\lambda_+ = \lambda_-$, then it is symmetric.
- $\lambda_+, \lambda_-$ and $\alpha$ are related to tail weights.
3 Lévy Process

3.1 Definitions and Properties

A cadlag (right-continuity and left limits) stochastic process \((X_t)_{t \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) with value in \(\mathbb{R}\) such that \(X_0 = 0\) is called a Lévy process if it possesses the following properties:

- Independent increments: for every increasing sequence of times \(t_0, \ldots, t_n\), the random variables \(X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}\) are independent.
- Stationary increments: \(X_{t+h} - X_t\) does not depend on \(t\).
- Stochastic continuity: \(\forall \epsilon > 0, \lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0\).

Assume that \(\psi : \mathbb{R} \to \mathbb{R}\) is a cumulant generating function of \(X_1\), then the characteristic function of a Lévy process \((X_t)_{t \geq 0}\) is

\[
E[e^{iuX_t}] = e^{t\psi(u)}, \quad u \in \mathbb{R}. \quad (3.1)
\]

Define Lévy measure \(\nu\) of \(X\) on \(\mathbb{R}\) as

\[
\nu(A) = E[\# \{t \in [0,1] : \triangle X_t \neq 0, \triangle X_t \in A\}], \quad A \in \mathcal{B}(\mathbb{R}) \quad (3.2)
\]

and \(\nu(A)\) is the expected number of jumps whose size belongs to \(A\) per unit time.

Based on Lévy-Khinchin representation, given \((X_t)_{t \geq 0}\) a Lévy process on \(\mathbb{R}\) with characteristic triplet \((\sigma^2, \nu, \gamma)\), then

\[
E[e^{iuX_t}] = \exp \left( i\gamma ut - \frac{\sigma^2 u^2}{2} t + t \int_{-\infty}^{\infty} (e^{iux} - 1 - iux \mathbb{1}_{|x| \leq 1}) \nu(dx) \right). \quad (3.3)
\]

Based on Lévy-Ito decomposition, a Lévy process can be decomposed as

\[
X_t = \sigma W_t + Z_t \quad (3.4)
\]

where \((W_t)_{t \geq 0}\) is a Brownian motion and \((Z_t)_{t \geq 0}\) is a pure jump process.

A Lévy process is of finite variation if and only if its characteristic triplet
\((\sigma^2, \nu, \gamma)\) satisfies
\[
\sigma = 0, \quad \int_{|x| \leq 1} |x| \nu(dx) < \infty. \tag{3.5}
\]

### 3.2 Infinitely Divisible Distributions

A random variable \(Y\) is referred as infinitely divisible if for each positive integer \(n\), there are independent and identically distributed random variables \(Y_1, Y_2, \cdots, Y_n\) such that \(Y = \sum_{k=1}^n Y_k\), i.e. the distribution of \(Y\) is the same as the distribution of \(\sum_{k=1}^n Y_k\).

We can prove that the Gaussian distribution, the gamma distribution, \(\alpha\)-stable distribution, the Poisson distribution and tempered stable distribution are infinitely divisible.

Let \((X_t)_{t \geq 0}\) be a Lévy process. Then for every \(t\), \(X_t\) has an infinitely divisible distribution. Conversely, if \(F\) is an infinitely divisible distribution then there exists a Lévy process \((X_t)_{t \geq 0}\) such that the distribution of \(X_1\) is given by \(F\).

Based on above proposition, infinitely divisible distributions can be used as building bricks for Lévy process and characterized by the Lévy measure \(\nu(dx)\).

- The Lévy measure of the \(\alpha\)-stable distribution is given by
  \[
  \nu_{\text{stable}}(dx) = \left( \frac{C_+}{x^{1+\alpha}} 1_{x>0} + \frac{C_-}{|x|^{1+\alpha}} 1_{x<0} \right) dx. \tag{3.6}
  \]

  The pure jump process \(X = (X_t)_{t \geq 0}\) is referred to as \(\alpha\)-stable process with parameters \((\alpha, \sigma, \beta, \mu)\), if \(X_1\) is the \(\alpha\)-stable random variable.

- The Lévy measure of general tempered distribution can be obtained by multiplying tempering function to the Lévy measure of \(\alpha\)-stable distribution. For example, given tempering function \(q(x) = e^{-\lambda_+ x} 1_{x>0} + e^{-\lambda_- |x|} 1_{x<0}\) we can get the Lévy measure of classical tempered stable distribution
  \[
  \nu(dx) = \left( \frac{C_+ e^{-\lambda_+ x}}{x^{1+\alpha}} 1_{x>0} + \frac{C_- e^{-\lambda_- |x|}}{|x|^{1+\alpha}} 1_{x<0} \right) dx. \tag{3.7}
  \]
3.3 Measure Transformation for Lévy Process

3.3.1 Equivalence of Measure for Lévy Process

As the Girsanov’s theorem for Brownian motion, with Esscher theorem[11] we can get the equivalent measure for Lévy process.

Suppose a process \( X = (X_t)_{t \geq 0} \) is a Lévy process with Lévy triplet \( (\sigma^2, \nu, \gamma) \) under measure \( \mathbb{P} \). If there is a real number \( \theta \) satisfying \( \int_{|x| \geq 1} e^{\theta x} \nu(dx) < \infty \), then we can find the equivalent measure \( Q \) whose Radon-Nikodym derivative is given by

\[
\frac{dQ}{d\mathbb{P}} |_{\mathcal{F}_t} = e^{\theta X_t} = e^{\theta X_t - l(\theta)t}
\]

where \( l(\theta) = \log E_{\mathbb{P}}[e^{\theta X_1}] \). Then

\[
Q(A) |_{\mathcal{F}_t} = \int_A \xi_t d\mathbb{P}, \quad A \in \mathcal{F}_t
\]

is equivalent to \( \mathbb{P}(A) |_{\mathcal{F}_t} \) for all \( t \geq 0 \). Moreover, the process \( X \) is a Lévy process with Lévy triplet \( (\sigma^2, \tilde{\nu}, \tilde{\gamma}) \) under measure \( Q \), where \( \tilde{\nu}(dx) = e^{\theta x} \nu(dx) \) and \( \tilde{\gamma} = \gamma + \int_{|x| \leq 1} x (e^{\theta x} - 1) \nu(dx) \).

Based on Esscher theorem, there is an equivalent proposition.

Suppose a process \( X = (X_t)_{t \geq 0} \) has Lévy triplet \( (\sigma^2, \nu, \gamma) \) under measure \( \mathbb{P} \) and Lévy triplet \( (\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma}) \) under measure \( Q \) if and only if three following conditions are satisfied:

- \( \sigma = \tilde{\sigma} \)
- \( \int_{-\infty}^{\infty} \left( e^{\psi(x)} - 1 \right) \nu(dx) < \infty \), where \( \psi(x) = \ln \left( \frac{d\tilde{\nu}}{d\nu} \right) \)
- If \( \sigma = 0 \), then we must in addition have

\[
\tilde{\gamma} - \gamma = \int_{|x| \leq 1} x (\tilde{\nu} - \nu)(dx)
\]

When \( \mathbb{P} \) and \( Q \) are equivalent, the Radon-Nikodym derivative is

\[
\frac{dQ}{d\mathbb{P}} |_{\mathcal{F}_t} = e^{\xi_t}
\]
where $\xi = (\xi_t)_{t \geq 0}$ is a Lévy process with Lévy triplet $(\sigma_\xi^2, \nu_\xi, \gamma_\xi)$ given by

$$\sigma_\xi^2 = \sigma^2 \eta$$  \hspace{1cm} (3.12)

$$\nu_\xi = \nu \psi^{-1}$$  \hspace{1cm} (3.13)

$$\gamma_\xi = -\frac{\sigma^2 \eta^2}{2} - \int_{-\infty}^{\infty} (e^{iy} - 1 - y 1_{|y| \leq 1}) \nu_\xi (dy)$$  \hspace{1cm} (3.14)

and $\eta$ is such that

$$\tilde{\gamma} - \gamma - \int_{|x| \leq 1} x (\tilde{\nu} - \nu) (dx) = \begin{cases} \sigma^2 \eta, & \sigma > 0 \\ 0, & \sigma = 0 \end{cases}.$$  \hspace{1cm} (3.15)

### 3.3.2 Change of Measure in Tempered Stable Process

We derive the equivalent measure of classical tempered stable distribution as an example for how to apply the Esscher transform to common Lévy process.

In the case of classical tempered stable case, we have the characteristic function

$$\phi_X_t (u) = \phi_{CTS} (u; \alpha, C, \lambda_+, \lambda_-, m)$$

$$= \exp \left( iu \gamma t - \sigma^2 u^2 t + t \int_{-\infty}^{\infty} (e^{iux} - 1 - ix 1_{|x| \leq 1}) \nu (dx) \right).$$  \hspace{1cm} (3.18)

and Lévy measure

$$\nu (dx) = \left( Ce^{-\lambda_+ x} 1_{x > 0} + Ce^{-\lambda_- |x|} 1_{x < 0} \right) dx.$$  \hspace{1cm} (3.17)

we need to derive Lévy triplet in order to apply the transform theorem.

Based on the definition of Lévy process which is defined by characteristic function

$$E \left[ e^{iuX_t} \right] = \exp \left( iu \gamma t - \frac{\sigma^2 u^2}{2} t + t \int_{-\infty}^{\infty} (e^{iux} - 1 - iux 1_{|x| \leq 1}) \nu (dx) \right).$$  \hspace{1cm} (3.18)
Then, we have
\[ \int_{-\infty}^{\infty} e^{ix} \nu(dx) = C \left( (\lambda_+ - iu)^\alpha + (\lambda_- + iu)^\alpha \right) \Gamma(-\alpha) \]  
(3.19)

\[ \int_{-\infty}^{\infty} \nu(dx) = \int_{0}^{\infty} \frac{Ce^{-\lambda_+ x}}{x^{1+\alpha}} dx + \int_{0}^{0} \frac{Ce^{-\lambda_- x}}{(-x)^{1+\alpha}} dx = C \left( \lambda_+^\alpha + \lambda_-^\alpha \right) \Gamma(-\alpha) \]  
(3.20)

\[ \int_{-\infty}^{\infty} iux 1_{|x| \leq 1} \nu(dx) \]
\[ = \int_{0}^{1} iux \frac{Ce^{-\lambda_+ x}}{x^{1+\alpha}} dx + \int_{-1}^{0} iux \frac{Ce^{-\lambda_- x}}{(-x)^{1+\alpha}} dx \]
\[ = iuC \left( \Gamma(1 - \alpha) \left( \lambda_+^{\alpha - 1} - \lambda_-^{\alpha - 1} \right) \right) \]  
(3.21)

and \( \sigma = 0, \gamma = m \).

Based on the Esscher theorem, we have
\[ l(\theta) = \log E_P \left[ e^{\theta X_1} \right] = \theta m - \theta \Gamma\left(1 - \alpha\right) \left( \lambda_+^{\alpha - 1} - \lambda_-^{\alpha - 1} \right) + \left(\lambda_+ - \theta\right)^\alpha - \left(\lambda_- + \theta\right)^\alpha - \lambda_-^\alpha \]  
(3.22)

then
\[ \tilde{\nu}(dx) = e^{\theta x} \nu(dx), \tilde{\gamma} = \gamma + \int_{-1}^{1} x \left( e^{\theta x} - 1 \right) \nu(dx) \]  
(3.23)

and the Radon-Nikodym derivative is
\[ \xi_t = e^{\theta X_t - l(\theta) t}. \]  
(3.24)

Thus we have the equivalent measure
\[ \tilde{C} = C, \tilde{\alpha} = \alpha, \tilde{\lambda}_+ = \lambda_+ - \theta, \tilde{\lambda}_- = \lambda_- + \theta \]

\[ \tilde{m} = m + \int_{-1}^{1} x \left( e^{\theta x} - 1 \right) \nu(dx). \]  
(3.25)

3.4 Exponential Lévy Model

3.4.1 General Definition of Exponential Lévy Model

Given a Lévy process \((X_t)_{t \geq 0}\), assume the stock price is defined by \(S_t = S_0 e^{X_t}\) at every time \(t\) and \(S_0\) is the initial value of the stock price, then the stock price follows an exponential Lévy model. The process \((S_t)_{t \geq 0}\) is referred to as the stock price process and \((X_t)_{t \geq 0}\) is referred as the driving process.
• If the driving process is Brownian motion, then the exponential Lévy model is referred as \textit{exponential Brownian motion model} or \textit{geometric Brownian motion model}. Assume $X_t = mt + \sigma W_t$, $m = \mu - \frac{\sigma^2}{2}$, then

$$E[S_t] = S_0 E[e^{X_t}] = S_0 e^{\mu t}. \quad (3.26)$$

• If the driving process is $\alpha$-stable process, then the exponential Lévy model is referred as \textit{exponential $\alpha$-stable process}.

![Figure 3.1: Exponential $\alpha$-Stable Model](image)
• If the driving process is tempered stable process, then the exponential Lévy model is referred as exponential tempered stable model, assume the tempered stable process with parameter \((\alpha, C, \lambda_+, \lambda_-, m)\) and \(m = \mu - \log(-i; \alpha, C, \lambda_+, \lambda_-, 0)\), then

\[
E[S_t] = S_0 E[e^{X_t}] = S_0 \exp(t \log(-i; \alpha, C, \lambda_+, \lambda_-, m))
\]

\[
= S_0 \exp(tm + t \log(-i; \alpha, C, \lambda_+, \lambda_-, 0)) = S_0 e^{\mu t}.
\] (3.27)

### 3.4.2 Absence of Arbitrage in Exponential Lévy Model

The Esscher transform can be used to construct equivalent martingale measure\(^{[12]}\) in exponential Lévy models.

Let \((X, \mathbb{P})\) be a Lévy process. If the trajectories of \(X\) are neither almost surely increasing nor almost surely decreasing, then the exponential Lévy model given by \(S_t = e^{rt + X_t}\) is arbitrage-free: there exists a probability measure \(\mathbb{Q}\) equivalent to \(\mathbb{P}\) such that \((e^{-rt} S_t)_{t \geq 0}\) is a \(\mathbb{Q}\)-martingale, where \(r\) is the interest rate.

In other words, the exponential Lévy model is arbitrage-free in the following cases:

- \(X\) has a nonzero Gaussian component: \(\sigma > 0\).
- \(X\) has infinite variation \(\int_{-1}^{1} |x| \nu(dx) = \infty\).
- \(X\) has both positive and negative jumps.
- \(X\) has positive jumps and negative drift or negative jumps and positive drift.
4 Option Pricing in Exponential Lévy Model

4.1 Non-Arbitrage Pricing

The portfolio which has the same cash flow as the payoff of a given option is referred to as the replicating portfolio of the option. Thus, according to the non-arbitrage pricing theory, the price of the option is same as the price of the portfolio.

A probability measure under which one cannot produce an arbitrage opportunity is referred to as the risk-neutral measure. If a probability measure $\mathbb{P}$ is estimated from historical return data for an underlying stock, the measure is referred to as the market measure or the physical measure. The risk-neutral measure can be found by the equivalent martingale measure (EMM) of the measure $\mathbb{P}$. A probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ is called an EMM of $\mathbb{P}$ if the discounted price process $\left(\tilde{S}_t = e^{-rt}S_t\right)_{t \in [0,T]}$ of underlying asset is a $\mathbb{Q}$-martingale, where $r$ is the risk-free rate of return. If the underlying asset has a continuous dividend rate $d$, then the discounted price process $\left(\tilde{S}_t = e^{-(r-d)t}S_t\right)_{t \in [0,T]}$ should be $\mathbb{Q}$-martingale.

The no-arbitrage price of a European option can be obtained by the expectation of the present value of the payoff for the options under the EMM $\mathbb{Q}$. That is, at time $t < T$, the non-arbitrage price of a European option $V_t$ with the payoff $\Pi(T)$ and the maturity $T$ is obtained by

$$V_t = e^{-r(T-t)}E_{\mathbb{Q}}[\Pi(T)|\mathcal{F}_t].$$

(4.1)

Given the risk neutral pricing rule, specifying an option pricing model is equivalent to specifying the law of $(S_t)_{t \geq 0}$ under $\mathbb{Q}$.

In Black-Scholes model, the risk-neutral dynamics of an asset price was described by the exponential of a Brownian motion with drift

$$S_t = S_0 \exp \left(B^0_t\right), \quad B^0_t = \left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t.$$  

(4.2)

Another formula obtained from Ito Lemma is

$$\frac{dS_t}{S_t} = rdt + \sigma dW_t = dB^1_t, \quad B^1_t = rt + \sigma W_t.$$  

(4.3)
A generalization can be obtained by simple replacement as

\[ S_t = S_0 \exp (rt + X_t) \]  

(4.4)
given

\[ \int_{|x| \geq 1} e^x \nu(dx) < \infty, \quad \gamma + \frac{\sigma^2}{2} + \int (e^y - 1 - y1_{|y| \leq 1}) \nu(dy) = 0 \]  

(4.5)
in order to guarantee that \( e^{-rt}S_t \) is a martingale.

Another formula can be given with a Lévy process \( Z_t \)

\[ dS_t = rS_t dt + S_t dZ_t. \]  

(4.6)

\( S_t \) then corresponds to the stochastic exponential of \( Z \).

### 4.2 Analytic Option Pricing Formula

The price of a call option can be expressed as the risk-neutral conditional expectation of the payoff

\[ c(t, S_t) = e^{-r(T-t)} E_Q \left[ (S_T - K)^+ | \mathcal{F}_t \right]. \]  

(4.7)

Given \( S_t = S_0 \exp (rt + X_t) \), we have

\[ S_T = S_t \exp [r(T-t) + (X_T - X_t)], \]  

(4.8)

thus we have the formula of call option price

\[ c(t, S_t) = e^{-r(T-t)} E_Q \left[ (S_t \exp [r(T-t) + (X_T - X_t)] - K)^+ \right]. \]  

(4.9)

Assume \( \tau = T - t \), then

\[
c(t, S_t) = e^{-r\tau} E_Q \left[ (S_t \exp [r\tau + X_\tau] - K)^+ \right]
\]

\[ = e^{-r\tau} \int_{-\infty}^{\infty} (S_t \exp [r\tau + x] - K)^+ f_{X_\tau}(x) \, dx \]

\[ = e^{-r\tau} \int_{-\infty}^{\infty} (S_t \exp [r\tau + x] - K)^+ \frac{1}{\pi} \text{Re} \left( \int_{0}^{\infty} e^{-iu\tau} \tilde{f}_{X_\tau}(u) \, du \right) \, dx \]

26
\[
\frac{e^{-r\tau} S_t}{\pi} \Re \int_{-\infty}^{\infty} \int_{0}^{\infty} \left( \exp (r \tau + x) - \frac{K}{S_t} \right)^+ \exp (-iu_x) \hat{\phi}_{X, \tau} (u) \, du \, dx
\]

\[
= \frac{e^{-r\tau} S_t}{\pi} \Re \int_{0}^{\infty} \int_{0}^{\infty} \frac{p}{p + K S_t} \exp (-iu \tau) \hat{\phi}_{X, \tau} (u) \, du \, dp \tag{4.10}
\]

4.3 Integro-Differential Equations and Numerical Methods

4.3.1 Integro-Differential Equations

In Black-Scholes model, where the risk neutral dynamics can be described by a diffusion process

\[
\frac{dS_t}{S_t} = rd t + \sigma dW_t \tag{4.11}
\]

the value of a European option can be computed by solving partial differential equation

\[
\frac{\partial C}{\partial t} (t, S) + rS \frac{\partial C}{\partial S} (t, S) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} (t, S) - rC (t, S) = 0 \tag{4.12}
\]

with boundary condition depending on the type of option.

A similar result can be obtained from Exponential Lévy model or a jump diffusion model: the value of a European option is given by \(C (t, S_t)\) which solves a second-order partial integro-differential equation (PIDE) [12]:

\[
\frac{\partial C}{\partial t} (t, S) + rS \frac{\partial C}{\partial S} (t, S) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} (t, S) - rC (t, S)
+ \int \nu (dy) \left[ C (t, Se^y) - C (t, S) - S (e^y - 1) \frac{\partial C}{\partial S} (t, S) \right] = 0 \tag{4.13}
\]

4.3.2 Finite Difference Methods

Assume \(x = \ln S\) and \(f (t, x) = C (t, S)\), then the formula can be transformed as

\[
\frac{\partial f}{\partial t} (t, x) + r \frac{\partial f}{\partial x} (t, x) + \frac{\sigma^2}{2} \left( \frac{\partial^2 f}{\partial x^2} (t, x) - \frac{\partial f}{\partial x} (t, x) \right) - rf (t, x)
+ \int \nu (dy) \left[ f (t, x + y) - f (t, x) - (e^y - 1) \frac{\partial f}{\partial x} (t, x) \right] = 0 \tag{4.14}
\]
Define the domain and the discrete grid on \([0, T] \times [-A, A]\):

\[ t_n = n\Delta t, \quad n = 0, \cdots, M, \quad \Delta t = \frac{T}{M}; \]

\[ x_i = -A + i\Delta x, \quad i = 0, \cdots, N, \quad \Delta x = \frac{2A}{N}. \] (4.15)

Let \(\{u^n_i\}\) be the numerical solution of above partial differential equation on the discretized grid, then we have the approximated formula

\[
\frac{\partial u}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} + \left( r - \frac{\sigma^2}{2} - \int \nu(dy) (e^{y} - 1) \right) \frac{\partial u}{\partial x} \\
- \left( r + \int \nu(dy) \right) u + \int \nu(dy) u(t, x + y) = 0. \] (4.16)

Given approximate boundaries \(B_l\) and \(B_r\), then

\[ \alpha = \int_{B_l}^{B_r} \nu(dy) (e^{y} - 1), \quad \beta = \int_{B_l}^{B_r} \nu(dy). \] (4.17)

Chosen \(K_l\) and \(K_r\), such that \([B_l, B_r] \subset [(K_l - \frac{1}{2}) \Delta x, (K_r + \frac{1}{2}) \Delta x]\)

\[ \int_{B_l}^{B_r} \nu(dy) u(t, x + y) \simeq \sum_{j=K_l}^{K_r} \nu_j u_{i+j}, \quad \beta \simeq \sum_{j=K_l}^{K_r} \nu_j, \]

\[ \alpha \simeq \sum_{j=K_l}^{K_r} \nu_j (e^{y_j} - 1), \quad \nu_j = \int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} \nu(dy). \] (4.18)

The space derivative can be discretized as

\[
\left( \frac{\partial^2 u}{\partial x^2} \right)_i \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}, \quad \left( \frac{\partial u}{\partial x} \right)_i \simeq \frac{u_{i+1} - u_{i-1}}{2\Delta x}. \] (4.19)

Assume \(a = r - \frac{\sigma^2}{2} - \alpha, \quad b = r + \beta\), we have explicit scheme

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{\sigma^2}{2} \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + a \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \\
- bu_i^n + \sum_{j=K_l}^{K_r} \nu_j u_{i+j} = 0. \] (4.20)
Then

\[ u_{i+1}^n = -\left( \frac{\sigma^2 \Delta t}{2 (\Delta x)^2} + \frac{a \Delta t}{2 \Delta x} \right) u_{i+1}^n - \left( \frac{\sigma^2 \Delta t}{2 (\Delta x)^2} - \frac{a \Delta t}{2 \Delta x} \right) u_{i-1}^n + \left( 1 + \frac{\sigma^2 \Delta t}{(\Delta x)^2} + b \Delta t \right) u_i^n - \sum_{j=K}^{K_r} \Delta t \nu_j u_{i+j}. \quad i = K_i, \ldots, N + K_r \quad (4.21) \]

This process can be summarized as matrix multiplication

\[
\vec{u}^n = \left[ u_{K_l}^n, u_{K_l+1}^n, \ldots, u_{N+K_r}^n, u_{N+K_r}^n \right]^T, \quad \vec{u}^{n+1} = A \vec{u}^n \quad (4.22)
\]

Let \( e_0 = \left( 1 + \frac{\sigma^2 \Delta t}{(\Delta x)^2} + b \Delta t - \Delta t \nu_0 \right) \), \( e_1 = \left( -\frac{\sigma^2 \Delta t}{2 (\Delta x)^2} - \frac{a \Delta t}{2 \Delta x} - \Delta t \nu_1 \right) \) and \( e_{-1} = \left( -\frac{\sigma^2 \Delta t}{2 (\Delta x)^2} + \frac{a \Delta t}{2 \Delta x} - \Delta t \nu_{-1} \right) \), we can denote the \((N + K_r - K_l + 1) \times (N + K_r - K_l + 1)\) matrix

\[
A = \begin{bmatrix}
    e_0 & e_1 & \Delta t \nu_2 & \cdots & \Delta t \nu_{N+2K_r} & 0 & 0 \\
    e_{-1} & e_0 & e_1 & \cdots & \cdots & 0 \\
    \Delta t \nu_{-2} & e_{-1} & e_0 & e_1 & \cdots & \Delta t \nu_{N+2K_r} \\
    \vdots & \Delta t \nu_{-2} & e_{-1} & e_0 & \cdots & \Delta t \nu_{2} \\
    \Delta t \nu_{2K_l} & \cdots & \cdots & \cdots & e_1 & \Delta t \nu_{2} \\
    0 & \cdots & \Delta t \nu_{-2} & e_{-1} & e_0 & e_1 \\
    0 & 0 & \Delta t \nu_{2K_l} & \cdots & \Delta t \nu_{-2} & e_{-1} & e_0 
\end{bmatrix} \quad (4.23)
\]
5 Future Work

5.1 Augmented Tempered Stable Groups

In the tempered stable groups, there are many members which have various properties and can satisfy different research requirements. Here we introduce some cases and expect to combine them with ARMA and GARCH models or high frequency analysis in further research.

5.1.1 Tempered Stable Groups

The generalized classical tempered stable distribution (GTS) has the characteristic function

\[ \phi_X(u) = \exp \left( ium - iu \Gamma(1 - \alpha) \left( C_+ \lambda_+^{\alpha+1} - C_- \lambda_-^{\alpha-1} \right) \right) \]

\[ + C_+ \Gamma(-\alpha+) \left( (\lambda_+ - iu)^{\alpha+} - \lambda_+^{\alpha+} \right) + C_- \Gamma(-\alpha-) \left( (\lambda_- + iu)^{\alpha-} - \lambda_-^{\alpha-} \right) \]  

where \( \alpha_+, \alpha_- \in (0, 1) \cup (1, 2) \), \( C_+, C_- > 0 \) and \( m \in \mathbb{R} \). This distribution can be denoted as \( X \sim GTS(\alpha_+, \alpha_-, C_+, C_-, \lambda_+, \lambda_-, m) \).

The modified tempered stable distribution (MTS) has the characteristic function

\[ \phi_X(u) = \exp (ium + C \left( G_R(u; \alpha, \lambda_+) + G_R(u; \alpha, \lambda_-) \right) + iuC \left( G_I(u; \alpha, \lambda_+) - G_I(u; \alpha, \lambda_-) \right) ) \]

where for \( u \in \mathbb{R} \)

\[ G_R(x; \alpha, \lambda) = 2^{-\frac{\alpha+3}{2}} \sqrt{\pi} \Gamma \left( -\frac{\alpha}{2} \right) \left( \lambda^2 + x^2 \right)^{\frac{\alpha}{2}} - \lambda^\alpha \]  

and

\[ G_I(x; \alpha, \lambda) = 2^{-\frac{\alpha+3}{2}} \Gamma \left( 1 - \frac{\alpha}{2} \right) \lambda^{\alpha-1} \left( F_1 \left( 1, \frac{1-\alpha}{2}; \frac{3}{2}; -\frac{x^2}{\lambda^2} \right) - 1 \right) \]

where \( F_1 \) is the hypergeometric function. This distribution can be denoted as \( X \sim MTS(\alpha_+, \alpha_-, C_+, C_-, \lambda_+, \lambda_-, m) \).

5.1.2 Rapidly Decreasing Tempered Stable Distribution

Given \( \alpha \in (0, 1) \cup (1, 2) \), \( C, \lambda_+, \lambda_- > 0 \) and \( m \in \mathbb{R} \), random variable \( X \) follows rapidly decreasing tempered stable (RDTS) distribution if the characteristic
function is
\[ \phi_X (u) = \phi_{\text{RDTS}} (u; \alpha, C, \lambda_+, \lambda_-, m) \]
\[ = \exp \left( iu m + C \left( G (iu; \alpha, \lambda_+) + G (-iu; \alpha, \lambda_-) \right) \right) \quad (5.5) \]

where
\[ G (x; \alpha, \lambda) = 2^{-\frac{\alpha}{2} - 1} \lambda^{-\alpha} \Gamma \left( \frac{\alpha}{2} \right) \left( M \left( -\frac{\alpha}{2}, \frac{1}{2}; \frac{x^2}{2\lambda^2} \right) - 1 \right) \]
\[ + 2^{-\frac{\alpha}{2} - \frac{1}{2}} \lambda^{-\alpha} x \Gamma \left( \frac{1 - \alpha}{2} \right) \left( M \left( \frac{1 - \alpha}{2}, \frac{3}{2}; \frac{x^2}{2\lambda^2} \right) - 1 \right) \quad (5.6) \]

and \( M \) is the confluent hypergeometric function which is the solution of the linear second-order differential equation
\[ xy'' + (c - x) y' - ay = 0. \]

The mean of \( X \) is \( m \) and the cumulants of \( X \) are
\[ c_n (X) = 2^{\frac{n - \alpha - 2}{2}} CT \left( \frac{n - \alpha}{2} \right) \left( \lambda^{\alpha-n}_+ + (-1)^n \lambda^{\alpha-n}_- \right), \quad n = 2, 3, \ldots \quad (5.7) \]

Assume
\[ C = 2^{\frac{\alpha}{2}} \left( \Gamma \left( \frac{n - \alpha}{2} \right) \left( \lambda^{\alpha-2}_+ + \lambda^{\alpha-2}_- \right) \right)^{-1} \quad (5.8) \]

then \( X \sim \text{RDTS} (\alpha, C, \lambda_+, \lambda_-, 0) \) is standard RDTS distribution has zero mean and unit variance.

### 5.2 Introduce ARMA and GARCH Option Pricing Model

Empirical studies shows that the amplitude of the daily returns varies across time. Volatility moves in clusters and we introduce ARMA and GARCH model to characterize this phenomenon.

**Autoregressive model** of order \( p \) can be denoted as AR\((p)\)
\[ X_t = c + \sum_{i=1}^{p} \varphi_i X_{t-i} + \epsilon_t \quad (5.9) \]

where \( \varphi_1, \ldots, \varphi_p \) are parameters, \( c \) is a constant and the random variable \( \epsilon_t \) is white noise.
Moving average model of order \( q \) can be denoted as \( \text{MA}(q) \)

\[
X_t = \mu + \epsilon_t + \sum_{i=1}^{q} \theta_i \epsilon_{t-i}
\]  

(5.10)

where \( \theta_1, \ldots, \theta_q \) are the parameters of the model, \( \mu \) is the expectation of \( X_t \) and \( \epsilon_t, \epsilon_{t-1}, \ldots, \epsilon_{t-q} \) are white noise error terms.

Autoregressive moving average model with \( p \) autoregressive terms and \( q \) moving average terms can be denoted as \( \text{ARMA}(p,q) \). This model contains the \( \text{AR}(p) \) and \( \text{MA}(q) \) models

\[
X_t = c + \sum_{i=1}^{p} \varphi_i X_{t-i} + \epsilon_t + \sum_{i=1}^{q} \theta_i \epsilon_{t-i}.
\]  

(5.11)

This general ARMA model is described by Peter Whittle[15] in 1951.

Autoregressive conditional heteroskedasticity (ARCH) models which is introduced by Engle (1982)[3] are used to characterize and model observed time series, in particular, assume the variance of current error term to be a function of the actual sizes of the previous time periods’ error terms.

ARCH models are employed commonly in modeling financial time series that exhibit time-varying volatility clustering. To estimate \( \text{ARCH}(q) \), given the autoregressive model \( \text{AR}(q) \)

\[
y_t = a_0 + a_1 y_{t-1} + \cdots + a_q y_{t-q} + \epsilon_t.
\]  

(5.12)

Assume \( \epsilon_t \) is the error term which can be split into a stochastic piece \( z_t \) and a time-dependent standard deviation \( \sigma_t \) characterizing the size of the terms

\[
\epsilon_t = \sigma_t z_t
\]  

(5.13)

the random variable \( z_t \) is a strong white noise process. The series \( \sigma_t^2 \) is modelled by

\[
\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2
\]  

(5.14)

where \( \alpha_0 > 0 \) and \( \alpha_i > 0, \ i > 0 \).

If an autoregressive moving average model is assumed for the error variance, the model is a \textit{generalized autoregressive conditional heteroskedasticity} (GARCH)[1] model.

The GARCH\((p,q)\) model (where \( p \) is the order of the GARCH terms \( \sigma^2 \) and
\( q \) is the order of the ARCH terms \( \epsilon^2 \) is given by

\[
\sigma_t^2 = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_q \epsilon_{t-q}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_p \sigma_{t-p}^2
\]

\[= \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-i}^2 \] (5.15)

Nonlinear GARCH (NGARCH) introduced by Engle and Ng (1993)[10] also known as Nonlinear Asymmetric GARCH(1,1) is introduced by Engle

\[
\sigma_t^2 = \omega + \alpha (\epsilon_{t-1} - \theta \sigma_{t-1})^2 + \beta \sigma_{t-1}^2
\] (5.16)

where \( \alpha, \beta \geq 0; \ \omega > 0. \)

5.3 High Frequency Analysis

5.3.1 High Frequency Data

The original form of market price is tick-by-tick data. By nature, these data are irregularly spaced in time. Liquid markets generate hundreds or thousands of ticks per business day. Thus high-frequency data should be the primary object of research [9].

Most published studies in the financial literature deal with low-frequency and regularly spaced data. With the development of electronic high-frequency trading, it is not only possible but also much more convenient to collect tick-by-tick data. Thus, we have to deal with vast amounts of data to better describe financial market.

5.3.2 Improve the Efficiency Current Algorithm

High-frequency trading had an execution time of several seconds, whereas by 2010 this had decreased to milliseconds and even microseconds. However, at the same time, we have to deal with vast amount of data. In sum, the efficiency of the algorithm is very important.

To improve the efficiency, we can introduce some advanced computing methods and skills, such as parallel computing.

5.4 Advanced Numerical Schemes and Application of Frontier

With the integro-differential equations, we can get the numerical solution
with different Lévy measure very efficiently. However, given the simplest explicit scheme discussed above, we can develop implicit or more accurate schemes. The stability, convergence and accuracy worth further analysis.

Based on the corresponding integro-differential equation of specific exponential Lévy model, we can explore abundant details of option prices with FronTier. Moreover, assisted by the front tracking function of FronTier, we can dig into some exotic options, such as American option, Asian option, Barrier option and so on.
References


