Introduction

Autoregressive Moving Average (ARMA) Model
Autoregressive Conditional Heteroskedastic Model
The GARCH Option Pricing Model

ARMA, GARCH and Related Option Pricing Method

Author: Yiyang Yang
Advisor: Pr. Xiaolin Li, Pr. Zari Rachev

Department of Applied Mathematics and Statistics
State University of New York at Stony Brook

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Time Series Model

Objective: forecasting, interpreting and testing hypotheses data

Method: decompose a series into a trend, seasonal, cyclical and irregular component

- these components can be represented as difference equation
- a difference equation expresses the value of a variable as a function of its own lagged values, time and other variables
- usually trend and seasonal term are both functions of time
- irregular term is a function of its own lagged value and stochastic variable

- many economic and finance theories have natural representation as stochastic differential equation
Figure: The Australian Red Wine Sales, Jan. 1980-Oct. 1991
The common type of difference equation is

\[ y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} + x_t \]

where \( x_t \) is the forcing process. And a special case for \( \{x_t\} \) is

\[ x_t = \sum_{i=0}^{\infty} \beta_i \epsilon_{t-i} \]

where \( \beta_i \) are constants and the individual elements \( \{\epsilon_t\} \) are independent of \( y_t \).
Solution by Iteration

- first-order difference equation $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$ with initial condition $y_0$
  have solution

$$y_t = a_0 \sum_{i=0}^{t-1} a_1^i + a_1^t y_0 + \sum_{i=0}^{t-1} a_1^i \epsilon_{t-i}$$

- the case without initial condition, substitute $y_0 = a_0 + a_1 y_{-1} + \epsilon_0$

$$y_t = a_0 \sum_{i=0}^{t-1} a_1^i + a_1^t y_{-1} + \sum_{i=0}^{t-1} a_1^i \epsilon_{t-i}$$

$$= a_0 \sum_{i=0}^{t+m} a_1^i + a_1^{t+m+1} y_{-m-1} + \sum_{i=0}^{t+m} a_1^i \epsilon_{t-i}$$

If $|a_1| < 1$, then $a_1^{t+m+1} y_{-m-1}$ approaches 0 as $m$ approaches $\infty$. Thus

$$y_t = \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \epsilon_{t-i}$$

However, the general solution is $y_t = A a_1^t + \frac{a_0}{1 - a_1} + \sum_{i=0}^{\infty} a_1^i \epsilon_{t-i}$
Lag Operators

Definition

The \textit{lag operator} $L$ is defined to be a \textit{linear} operator such that for any value $y_t$

\[ L^i y_t = y_{t-i}. \]

- the lag of a constant is a constant $Lc = c$
- distributive law $(L^i + L^j) y_t = L^i y_t + L^j y_t = y_{t-i} + y_{t-j}$
- associative law $L^i L^j y_t = L^i (L^j y_t) = L^i y_{t-j} = y_{t-i-j}$
- $L$ raised to a negative power is a \textit{lead factor} $L^{-i} y_t = y_{t+i}$
- For $|a| < 1$, $(1 + aL + a^2L^2 + \cdots) y_t = \frac{y_t}{1-aL}$
- the $p$th-order equation $y_t = a_0 + a_1 y_{t-1} + \cdots + a_p y_{t-p} + \epsilon_t$ can be written

\[ (1 - a_1 L - \cdots - a_p L^p) y_t = A(L) y_t = a_0 + \epsilon_t \]
Definition

The sequence \( \{\epsilon_t\} \) is a white-noise process if for each time period \( t \)

\[
E(\epsilon_t) = E(\epsilon_{t-1}) = \cdots = 0
\]

\[
E(\epsilon_t^2) = E(\epsilon_{t-1}^2) = \cdots = \sigma^2
\]

\[
\text{or } \text{var}(\epsilon_t^2) = \text{var}(\epsilon_{t-1}^2) = \cdots = \sigma^2
\]

and for all \( j \) and all \( s \)

\[
E(\epsilon_t \epsilon_{t-s}) = E(\epsilon_{t-j} \epsilon_{t-j-s}).
\]
The moving average model of order $q$ denoted as $MA(q)$ can be written as
\[ x_t = \sum_{i=0}^{q} \beta_i \epsilon_{t-i} \]
where $\beta_0, \ldots, \beta_q$ are the parameters of the model and $\{\epsilon_t\}$ are white noise.

The autoregressive model of order $p$ denoted as $AR(p)$ which can be written as
\[ y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \epsilon_t \]
where $a_1, \ldots, a_p$ are parameters, $a_0$ is a constant and the random variable $\{\epsilon_t\}$ are white noise.
Definition

Combine moving average model with autoregressive model, we get *autoregressive moving average* (ARMA) model denoted as ARMA\((p, q)\)

\[ y_t = a_0 + \sum_{i=1}^{p} a_i y_{t-i} + \epsilon_t + \sum_{i=1}^{q} \beta_i \epsilon_{t-i} \]

if the characteristic roots of the equation are all in unit circle.

- If one or more characteristic roots is greater than or equal to unity, the sequence \(\{y_t\}\) is an *integrated process* and is called an *autoregressive integrated moving average* (ARIMA) model.
- Rewrite the model with lag operators

\[
\left(1 - \sum_{i=1}^{p} a_i L^i\right) y_t = a_0 + \sum_{i=0}^{q} \beta_i \epsilon_{t-i}
\]

then the particular solution is

\[
y_t = \frac{a_0 + \sum_{i=0}^{q} \beta_i \epsilon_{t-i}}{1 - \sum_{i=1}^{p} a_i L^i}.
\]
Stationary

Definition

A stochastic process \( \{y_t\} \) is stationary if for all \( t \) and \( t - s \)

\[
E(y_t) = E(y_{t-s}) = \mu
\]

\[
E\left[(y_t - \mu)^2\right] = E\left[(y_{t-s} - \mu)^2\right] = \sigma^2
\]

\[
E[(y_t - \mu)(y_{t-s} - \mu)] = E[(y_{t-j} - \mu)(y_{t-j-s} - \mu)] = \gamma_s
\]

where \( \mu, \sigma^2, \gamma_s \) are all constants. \( \gamma_s \) is autocovariance function (ACF).

For stationary process \( \{y_t\} \), the mean, variance and autocorrelations can be well approximated by sufficiently long time averages based on the single set of realization.
Stationary Restrictions for ARMA(p,q) (Cont.)

- Consider $MA(\infty)$ model $x_t = \sum_{i=0}^{\infty} \beta_i \epsilon_{t-i}$
  - Expectation
    $$E(x_t) = E(\epsilon_t + \beta_1 \epsilon_{t-1} + \beta_2 \epsilon_{t-2} + \cdots) = 0$$
    constant and independent of $t$.
  - Variance
    $$Var(x_t) = E[(\epsilon_t + \beta_1 \epsilon_{t-1} + \beta_2 \epsilon_{t-2} + \cdots)^2] = \sigma^2 \left(1 + \beta_1^2 + \beta_2^2 + \cdots\right)$$
    constant and independent of $t$.
  - Covariance
    $$E(x_t x_{t-s}) = E[(\epsilon_t + \beta_1 \epsilon_{t-1} + \cdots)(\epsilon_{t-s} + \beta_1 \epsilon_{t-s-1} + \cdots)]$$
    $$= \sigma^2 (\beta_s + \beta_1 \beta_{s+1} + \beta_2 \beta_{s+2} + \cdots)$$
    only depend on $s$ and the value should be finite.

- By constraint $\sum_{i=0}^{\infty} \beta_i^2 < \infty$, $ARMA(p,q)$ can be transformed to $MA(\infty)$. 

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The autocorrelation function (ACF) of \( y_t \) and \( y_{t-s} \) is defined by autocovariance function \( \gamma_s \) as
\[
\rho_s = \frac{\gamma_s}{\gamma_0}.
\]

Consider \( ARMA(1,1) \) case \( y_t = a_1 y_{t-1} + \epsilon_t + \beta_1 \epsilon_{t-1} \), then Yule-Walker equations are:
\[
Ey_t y_t = a_1 Ey_{t-1} y_t + E \epsilon_t y_t + \beta_1 E \epsilon_{t-1} y_t \Rightarrow \gamma_0 = a_1 \gamma_1 + \sigma^2 + \beta_1 (a_1 + \beta_1) \sigma^2
\]
\[
Ey_t y_{t-1} = a_1 Ey_{t-1} y_{t-1} + E \epsilon_t y_{t-1} + \beta_1 E \epsilon_{t-1} y_{t-1} \Rightarrow \gamma_1 = a_1 \gamma_0 + \beta_1 \sigma^2
\]
\[
Ey_t y_{t-s} = a_1 Ey_{t-1} y_{t-s} + E \epsilon_t y_{t-s} + \beta_1 E \epsilon_{t-1} y_{t-s} \Rightarrow \gamma_s = a_1 \gamma_{s-1}, \ s = 2, 3, \ldots
\]
Solve first two equations, we get

$$\gamma_0 = \frac{1 + \beta_1^2 + 2a_1\beta_1}{1 - a_1^2}\sigma^2, \quad \gamma_1 = \frac{(1 + a_1\beta_1)(a_1 + \beta_1)}{1 - a_1^2}\sigma^2$$

hence, we get autocorrelation function

$$\rho_1 = \frac{(1 + a_1\beta_1)(a_1 + \beta_1)}{1 + \beta_1^2 + 2a_1\beta_1}$$

For all $s \geq 2$, given $\gamma_s = a_1\gamma_{s-1}$, we have

$$\rho_s = a_1\rho_{s-1}.$$ 

If $0 < a_1 < 1$, ACF converges directly; if $-1 < a_1 < 0$, ACF will oscillate.
Autocorrelation Function (Cont.)

**Figure:** The ACF of $AR(2), a_1 = 2, a_2 = 5$ and $AR(2), a_1 = 10/9, a_2 = 2$
Figure: The ACF of $AR(2)$, $a_1 = -\frac{10}{9}$, $a_2 = 2$ and $AR(2)$, $a_1 = \frac{2(1+i\sqrt{3})}{3}$, $a_2 = \frac{2(1-i\sqrt{3})}{3}$
The *partial autocorrelations function* (PACF) can be obtained from ACF

\[ \phi_{11} = \rho_1, \quad \phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} \]

\[ \phi_{ss} = \frac{\rho_s - \sum_{j=1}^{s-1} \phi_{s-1} \rho_s-j}{1 - \sum_{j=1}^{s-1} \phi_{s-1} \rho_j}, \quad s = 3, 4, 5 \ldots \]

where \( \phi_{sj} = \phi_{s-1,j} - \phi_{ss} \phi_{s-1,s-j}, \quad j = 1, 2, \ldots, s - 1. \)

- partial autocorrelation ignores the effect of the intervening values
- PACF plays an important role in identifying the lag in an AR model
- direct method for estimating PACF:
  - form series \( \{y_t^*\} \) by subtracting the mean of \( \{y_t\} \):  \( y_t^* = y_t - \mu \)
  - form autoregression equation
    \[ y_t^* = \phi_{s1} y_{t-1}^* + \phi_{s2} y_{t-2}^* + \cdots + \phi_{ss} y_{t-s}^* + e_t \]
  - \( \phi_{ss} \) is the partial autocorrelation coefficient between \( y_t \) and \( y_{t-s} \)
Partial Autocorrelation Function (Cont.)

Figure: ACF and corresponding PACF of specific ARMA model
Empirical Facts

- Most of the series contains a clear trend.
- Any shock to a series displays a high degree of persistence.
- The volatility of many series is not constant over time.
  - A stochastic variable with a constant variance is called *homoskedastic* as opposed to *heteroskedastic*.
  - *Conditionally heteroskedastic*: unconditional (or long-term) variance is constant but there are periods with relative high variance.
- Some series share comovements with other series.
Empirical Facts

Four figures denote: GDP and actual Consumption, Federal funds and 10-year bond, NYSE stock index and Exchange rate of US, Canada and UK

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For investor plan to buy at $t$ and sell at $t + 1$, forecasts of the rate of return and variance over the holding period is more important than unconditional variance.

Simple example

$$y_{t+1} = \epsilon_{t+1}x_t$$

where $y_{t+1}$ is variable of interest, $\epsilon_{t+1}$ is white-noise with variance $\sigma^2$ and $x_t$ is an independent variable observed at period $t$.

- If $x_t = x_{t-1} = \cdots = constant$, $\{y_t\}$ is a white-noise process.
- If $\{x_t\}$ are not all equal, then

$$Var \left( y_{t+1} | x_t \right) = x_t^2 \sigma^2$$

- If $x_t^2$ is large (small), the variance of $y_{t+1}$ will be large (small).
- Introduction of $\{x_t\}$ can explain periods of volatility in $\{y_t\}$.
Preliminary idea of Engle: conditional forecasts are vastly superior to unconditional forecasts.

Given stationary ARMA model $y_t = a_0 + a_1 y_{t-1} + \epsilon_t$, the conditional forecasts are

$$E_t y_{t+1} = a_0 + a_1 y_t, \quad E_t \left[ (y_{t+1} - a_0 - a_1 y_1)^2 \right] = E_t \epsilon_{t+1}^2 = \sigma^2$$

the unconditional forecasts are

$$Ey_{t+1} = \frac{a_0}{1 - a_1}, \quad E \left[ \left( y_{t+1} - \frac{a_0}{1 - a_1} \right)^2 \right] = \frac{\sigma^2}{1 - a_1^2}$$

Since $\frac{\sigma^2}{1 - a_1^2} > \sigma^2$, unconditional forecast has greater variance than conditional forecast.

Conditional forecasts consider both current and past realization of series are preferable.
Suppose the conditional variance is not a constant but an $AR\ (q)$ process, we obtain an \textit{autoregressive conditional heteroskedastic} (ARCH) model for the square of the estimated residuals:

$$\hat{\epsilon}_t^2 = \alpha_0 + \alpha_1 \hat{\epsilon}_{t-1}^2 + \alpha_2 \hat{\epsilon}_{t-2}^2 + \cdots + \alpha_q \hat{\epsilon}_{t-q}^2 + \nu_t$$

where $\nu_t$ is a white noise process.

- If $\alpha_1, \alpha_2, \cdots, \alpha_n$ all equal zero, the estimated variance is constant.
- There are many possible applications for ARCH model, since the residuals can come from an autoregression, an ARMA model or a standard regression model.
- The linear ARCH model is not most convenient. The model for $\{y_t\}$ and the conditional variance are best estimated simultaneously using maximum likelihood techniques. A better way is to specify $\nu_t$ as a multiplicative disturbance.
The simplest example from the class of multiplicative conditionally heteroskedastic models proposed by Engle (1982) is

\[ \epsilon_t = v_t \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2} \]

where \( v_t \) is white-noise process with \( \sigma_v^2 = 1 \) and \( \epsilon_{t-1} \) are mutually independent, \( \alpha_0 > 0 \) and \( 0 < \alpha_1 < 1 \) are constants.

- Given \( Ev_t = 0 \), the unconditional expectation is
  \[ E\epsilon_t = E \left[ v_t \left( \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \right)^{1/2} \right] = Ev_tE \left( \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \right)^{1/2} = 0 \]

- Given \( Ev_t v_{t-i} = 0 \)
  \[ E\epsilon_t \epsilon_{t-i} = 0, \ i \neq 0 \]

- The unconditional variance is
  \[ E\epsilon_t^2 = E \left[ v_t^2 \left( \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \right) \right] = Ev_t^2 E \left( \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \right) \]

since \( Ev_t^2 = \sigma^2 = 1 \) and unconditional variance of \( \epsilon_t \) is identical to \( \epsilon_{t-1} \), we have

\[ E\epsilon_t^2 = \frac{\alpha_0}{1 - \alpha_1} \]
The conditional mean of $\epsilon_t$ is

$$E(\epsilon_t | \epsilon_{t-1}, \epsilon_{t-2}, \cdots) = Ev_t E \left( \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \right)^{1/2} = 0$$

The conditional variance of $\epsilon_t$ is

$$E \left( \epsilon_t^2 | \epsilon_{t-1}, \epsilon_{t-2}, \cdots \right) = \alpha_0 + \alpha_1 \epsilon_{t-1}^2$$

The multiplicative form has no influence on the unconditional expectation and variance, conditional mean.

The influence falls entirely on conditional variance which is a first-order autoregressive process denoted by $ARCH(1)$.

The higher order $ARCH(q)$ processes

$$\epsilon_t = \nu_t \sqrt{\alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2}$$

since $\epsilon_{t-1}$ to $\epsilon_{t-q}$ have a direct effect on $\epsilon_t$, thus conditional variance acts like an autoregressive process of order $q$. 
Bollerslev (1986) extended Engle’s work by introducing both autoregressive and moving average components (ARMA) in heteroskedastic variance which is called generalized ARCH \((p, q)\) model or GARCH \((p, q)\) model.

\[
\epsilon_t = \nu_t \sqrt{h_t}
\]

where \(\sigma^2_v = 1\) and

\[
h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}
\]

- The key feature of GARCH model is that the conditional variance of the disturbances of \(\{y_t\}\) sequence constitutes an ARMA process.

- The conditional variance is

\[
E_t \epsilon_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}
\]
Introduction

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Expanded GARCH Model

Definition

The Nonlinear GARCH (NGARCH) model is also known as Nonlinear Asymmetric GARCH (1, 1) was introduced by Engle and Ng in 1993

$$h_t = \omega + \alpha (\epsilon_{t-1} - \theta h_{t-1})^2 + \beta h_{t-1}^2, \quad \omega, \alpha, \beta \geq 0.$$  

Definition

Integrated GARCH (IGARCH) is a restricted version of GARCH model, with condition

$$\sum_{i=1}^{p} \beta_i + \sum_{i=1}^{q} \alpha_i = 1.$$  

Definition

The GARCH-in-mean (GRACH-M) model adds a heteroskedasticity term to mean

$$y_t = \beta x_t + \delta h_t + \epsilon_t, \quad \epsilon_t = \nu_t \sqrt{h_t}, \quad h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2.$$  

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ARMA, GARCH and Related Option Pricing Method
GARCH option pricing model has three distinctive features:

- The GARCH option pricing is a function of *risk premium* embedded in the underlying asset.
  - standard preference-free option
- GARCH option pricing model is *non-Markovian*.
  - the underlying asset value is a diffusion process
  - standard approach is Markovian
- GARCH option pricing can potentially explain some well-documented *systematic biases* associated with Black-Scholes model.
  - underpricing of short-maturity option
  - implied volatility smile
  - underpricing of options on low-volatility securities
Consider discrete-time economy and let \( X_t \) be the asset price at time \( t \).

One-period of return under probability measure \( P \) is

\[
\ln \frac{X_t}{X_{t-1}} = r + \lambda \sqrt{h_t} - \frac{1}{2} h_t + \epsilon_t
\]

where \( \epsilon_t \) has mean zero and conditional variance \( h_t \), \( r \) is the constant one-period risk-free rate of return and \( \lambda \) the constant unit risk premium.

Assume that \( \epsilon_t \) follows a \( GARCH(p, q) \) under measure \( P \)

\[
\epsilon_t | \phi_{t-1} \sim N(0, h_t)
\]

\[
h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i}.
\]

If \( p = 0 \) and \( q = 0 \), this is exactly the Black-Scholes case.
The conventional risk-neutral valuation relationship has to be generalized to accommodate heteroskedasticity of the asset return process.

**Definition**

A pricing measure $Q$ is said to satisfy the locally risk-neutral valuation relationship (LRNVR) if measure $Q$ is mutually absolutely continuous with respect to measure $P$, $X_t/X_{t-1} | \phi_{t-1}$ distributes lognormally (under $Q$)

$$E^Q (X_t/X_{t-1} | \phi_{t-1}) = e^r,$$

$$\text{Var}^Q (\ln (X_t/X_{t-1}) | \phi_{t-1}) = \text{Var}^P (\ln (X_t/X_{t-1}) | \phi_{t-1})$$

almost surely with respect to measure $P$. 
The LRNVR implies that, under pricing measure $Q$

\[ \ln \frac{X_t}{X_{t-1}} = r - \frac{1}{2} h_t + \xi_t \]

where

\[ \xi_t | \phi_{t-1} \sim N(0, h_t) \]

\[ h_t = \alpha_0 + \sum_{i=1}^{q} \alpha_i \left( \xi_{t-i} - \lambda \sqrt{h_{t-i}} \right)^2 + \sum_{i=1}^{p} \beta_i h_{t-i}. \]

- The asset price can be denoted as

\[ X_T = X_t \exp \left[ (T - t) r - \frac{1}{2} \sum_{s=t+1}^{T} h_s + \sum_{s=t+1}^{T} \xi_s \right] \]

- The discounted asset price process $e^{-rt}X_t$ is $Q$-martingale.
Theorem

Under the GARCH \((p, q)\) specification, a European option with exercise price \(K\) maturing at time \(T\) has time \(t\) value to

\[
C_t^{GH} = e^{-(T-t)r}E^Q \left[ \max (X_T - K, 0) \mid \phi_t \right].
\]

Corollary

The delta of an option is the first partial derivative of the option price with respect to the underlying asset price. The GARCH option delta at time \(t\) by \(\Delta_t^{GH}\) is

\[
\Delta_t^{GH} = e^{-(T-t)r}E^Q \left[ \frac{X_T}{X_t} 1_{\{X_T \geq K\}} \mid \phi_t \right].
\]
Definition

The log-returns of asset are assumed to follow the following dynamic under the objective measure $P$

$$\log \left( \frac{X_t}{X_{t-1}} \right) = r_t - d_t + \lambda_t \sqrt{h_t} - g \left( \sqrt{h_t}; \theta \right) + \epsilon_t$$

$$\epsilon_t = \sqrt{h_t} v_t, \quad h_t = \alpha_0 + \alpha_1 h_{t-1} (\epsilon_{t-1} - \gamma)^2 + \beta_1 h_{t-1}$$

where $X_t$ is closing price of underlying asset at date $t$, $r_t$ and $d_t$ denote continuously compounded risk-free rate of return and dividend rate for $[t - 1, t]$. $v_t$, with parameter $\theta$, is a random variable with zeros mean and unit variance. $g \left( x; \theta \right) = \log \left( \mathbb{E} [e^{xv_t}] \right)$ is the log-Laplace-transform of $v_t$.

- If the distribution of $v_t$ is standard normal distribution, then this is Duan’s option pricing model.
- For $\alpha_1 = \beta_1 = 0$, this is discrete-time Black-Scholes log-normal model with constant volatility.
Theorem

For any standardized probability distribution $\xi_t$ with parameters $\tilde{\theta}$ which is equivalent to the marginal distribution of $v_t$, then the distribution of following process is equivalent to the NGARCH stock price model described above

$$\log \left( \frac{X_t}{X_{t-1}} \right) = r_t - d_t - g \left( \sqrt{h_t}; \tilde{\theta} \right) + \sqrt{h_t} \xi_t$$

$$h_t = \alpha_0 + \alpha_1 h_{t-1} \left( \xi_{t-1} - \lambda_t + \frac{1}{\sqrt{h_{t-1}}} \left( g \left( \sqrt{h_{t-1}}; \tilde{\theta} \right) - g \left( \sqrt{h_{t-1}}; \theta \right) \right) - \gamma \right)^2 + \beta_1 h_{t-1}.$$ 

- This theorem allows for different distribution forcing process under the objective measure and the risk-neutral measure.
- A discrete-time financial market with a continuous distribution for the return is inevitably incomplete.
- This is just a possible risk-neutral measure.
References