Stable Distribution: Theory and Application

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2 Stable Distribution
   - Definition and Properties of Stable Distribution
   - Estimation of Parameters
   - Simulation of $\alpha$-Stable Random Variable

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4 Brief Introduction to Advanced Topics
   - Tempered Stable Distribution
   - Lévy Process and Exponential Lévy Model
   - Partial-Integro Differential Equation for Option Pricing
**Background**

**Traditional Assumption**
- Louis Bachelier 1900
  - log return of underlying asset is Gaussian distribution
  - prices behave like random walk
  - widely applied, such as Black-Scholes model
  - central limit theorem

**Stable Assumption**
- Mandelbrot (1962) and Fama (1965)
  - study the empirical distribution of returns on common stocks
  - propose *Stable Paretian distribution*
  - excess kurtosis
  - generalized central limit theorem
Motivation

Empirical Observations

- **Heavy tail**
  - tail: where extreme values occur

- **Skewness**
  - non-symmetric
  - one tail is longer than the other one

- **Volatility Clustering**
  - tendency of large changes to be followed by large changes
Outline

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Definition

Given independent and identically distributed random variables $X_1, X_2, \cdots, X_n$ and $X$, then $X$ is said to follow an $\alpha$-Stable distribution if there exists a positive constant $C_n$ and a real number $D_n$ such that the following relation holds:

$$X_1 + X_2 + \cdots + X_n \equiv C_nX + D_n$$

where $\equiv$ denotes equality in distribution.

- When $\alpha = 2$, it is the Gaussian case; when $0 < \alpha < 2$, we have the non-Gaussian case.
There are three special cases with a closed form probability density function:

- **the Gaussian case** ($\alpha = 2$)

  \[
  f(x) = \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(x-\mu)^2}{4\sigma^2}}
  \]

- **the Cauchy case** ($\alpha = 1, \beta = 0$)

  \[
  f(x) = \frac{\sigma}{\pi ((x - \mu)^2 + \sigma^2)}
  \]

- **the Lévy case** ($\alpha = \frac{1}{2}, \beta = \pm 1$)

  \[
  f(x) = \frac{\sqrt{\sigma}}{\sqrt{2\pi} (x - \mu)^{\frac{3}{2}}} e^{-\frac{\sigma}{2(x-\mu)}}
  \]
\( \alpha \)-Stable distribution does not have closed form density function and is expressed by characteristic function:

\[
\phi_{\text{stable}} (t; \alpha, \sigma, \beta, \mu) = E \left[ e^{itX} \right] = \begin{cases} 
\exp \left( i\mu t - |\sigma t|^\alpha \left( 1 - i\beta (\text{sign} t) \tan \frac{\pi \alpha}{2} \right) \right) & \alpha \neq 1 \\
\exp \left( i\mu t - \sigma |t| \left( 1 + i\beta \frac{2}{\pi} (\text{sign} t) \ln |t| \right) \right) & \alpha = 1
\end{cases}
\]

where

\[
\text{sign} t = \begin{cases} 
1, & t > 0 \\
0, & t = 0 \\
-1, & t < 0
\end{cases}
\]

Four related parameters are:

- \( \alpha \): the index of stability or the shape parameter, \( \alpha \in (0, 2) \)
- \( \beta \): the skewness parameter, \( \beta \in [-1, 1] \)
- \( \sigma \): the scale parameter, \( \sigma \in (0, +\infty) \)
- \( \mu \): the location parameter, \( \mu \in (-\infty, +\infty) \)
Definition (Cont.)

**Stable Distribution: Theory and Application**

- **Introduction**
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- **Algorithms and Numerical Results**
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**Definition and Properties of Stable Distribution**

**Estimation of Parameters**

**Simulation of α-Stable Random Variable**

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**Yi Yang Yang**

**Stable Distribution: Theory and Application**
Definition (Cont.)

Stable Distribution: Theory and Application

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Properties of $\alpha$-Stable Distribution

There are several basic properties of $\alpha$-stable distribution:

- The tail of the density function decays like a power function
  
  \[ P \left( |X| > x \right) \propto C \cdot x^{-\alpha}, \, x \to \infty \]

  for some constant $C$.

- Raw moments satisfy the property
  
  \[ E |X|^p < \infty, \, 0 < p < \alpha \]

  \[ E |X|^p = \infty, \, p \geq \alpha \]

- The expectation of $\alpha$-stable distribution is
  
  \[ E [X] = \mu, \, \alpha > 1 \]

  \[ E [X] = \infty, \, \alpha \leq 1 \]
The stability property is preserved under linear transformation. Suppose \( \{X_i\} \) are i.i.d. random variable such that \( X_i \sim S_\alpha (\sigma_i, \beta_i, \mu_i) \), then

- \( Y = \sum_{i=1}^{n} X_i \) is an \( \alpha \)-stable distribution with stability index \( \alpha \) and parameters
  \[
  \beta = \frac{\sum_{i=1}^{n} \beta_i \sigma_i^\alpha}{\sum_{i=1}^{n} \sigma_i^\alpha}, \quad \sigma = \left( \sum_{i=1}^{n} \sigma_i^\alpha \right)^{\frac{1}{\alpha}}, \quad \mu = \sum_{i=1}^{n} \mu_i
  \]

- \( Y = X_1 + a \) is an \( \alpha \)-stable distribution with stability index \( \alpha \) and parameters
  \[
  \beta = \beta_1, \quad \sigma = \sigma_1, \quad \mu = \mu_1 + a
  \]

- \( Y = aX_1, a \neq 0 \) is an \( \alpha \)-stable distribution with stability index \( \alpha \) and parameters
  \[
  \beta = (\text{sign } a) \beta_1, \quad \sigma = |a| \sigma_1, \quad \mu = \begin{cases} 
  a\mu_1 & \alpha \neq 1 \\
  a\mu_1 - \frac{2}{\pi} a (\ln |a|) \sigma_1 \beta_1 & \alpha = 1
  \end{cases}
  \]
Definition

Given sample of observed data $X = \{x_1, x_2, \cdots, x_N\}$ and assume it is directed by $\alpha$-stable distribution. Then, define the sample characteristic function

$$\hat{\phi}_X (u) = \frac{1}{N} \sum_{j=1}^{N} e^{iux_j}.$$ 

- By the law of large number, $\hat{\phi}_X (u)$ is a consistent estimator of the characteristic function $\phi_X (u)$.
- By simple transformation, we have for all $\alpha$

$$|\phi_X (u)| = \exp (-\sigma^\alpha |u|^\alpha).$$

Thus

$$-\log |\phi_X (u)| = \sigma^\alpha |u|^\alpha.$$ 

Assume $\alpha \neq 1$, choose two different nonzero values $u_k$, $k = 1, 2$, then

$$-\log |\hat{\phi}_X (u_k)| = \sigma^\alpha |u_k|^\alpha.$$
Sample Characteristic Function Method (Cont.)

- Solve these two equations and get $\hat{\alpha}, \hat{\sigma}$

$$
\hat{\alpha} = \log \frac{\log |\hat{\phi}(u_1)|}{\log |\hat{\phi}(u_2)|}
$$

$$
\log \hat{\sigma} = \frac{\log |u_1| \log \left( - \log |\hat{\phi}(u_2)| \right) - \log |u_2| \log \left( - \log |\hat{\phi}(u_1)| \right)}{\log \left| \frac{u_1}{u_2} \right|}.
$$

- The estimation of $\hat{\beta}$ and $\hat{\mu}$ based on the imaginary and real parts of the characteristic function

$$
Re (\phi_X (u)) = \exp (- |\sigma \mu|^\alpha) \cos \left( \mu u + |\sigma u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2} \right),
$$

$$
Im (\phi_X (u)) = \exp (- |\sigma \mu|^\alpha) \sin \left( \mu u + |\sigma u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2} \right).
$$

Then, we have

$$
\left( \arctan \frac{\text{Im}(\phi_X (u))}{\text{Re}(\phi_X (u))} \right) = \mu u + |\sigma u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2}.
$$
Based on $\hat{\alpha}$, $\hat{\sigma}$ and two different nonzero values $u_k$, $k = 3, 4$, we can solve the system of equations to obtain estimation of $\hat{\beta}$ and $\hat{\mu}$.

$$\hat{\mu} = \frac{u_4 \arctan \frac{\text{Im}(\phi_X(u_3))}{\text{Re}(\phi_X(u_3))} - u_3 \arctan \frac{\text{Im}(\phi_X(u_4))}{\text{Re}(\phi_X(u_4))}}{u_3 u_4^\hat{\alpha} - u_4 u_3^\hat{\alpha}}$$

$$\hat{\beta} = \frac{u_4 \arctan \frac{\text{Im}(\phi_X(u_3))}{\text{Re}(\phi_X(u_3))} - u_3 \arctan \frac{\text{Im}(\phi_X(u_4))}{\text{Re}(\phi_X(u_4))}}{\hat{\sigma}^\hat{\alpha} \tan \frac{\pi \hat{\alpha}}{2} \left( u_4 u_3^\hat{\alpha} - u_3 u_4^\hat{\alpha} \right)}.$$

In this estimation, the values are $u_1 = 0.2$, $u_2 = 0.8$, $u_3 = 0.1$ and $u_4 = 0.4$ are proposed in the simulation study.
An advanced application of sample characteristic function is *time regression*.

Based on the equations

\[
\log \left( -\log |\phi_X(u)|^2 \right) = \log 2\sigma^\alpha + \alpha \log |u|
\]

\[
\left( \arctan \frac{\text{Im}(\phi_X(u))}{\text{Re}(\phi_X(u))} \right) = \mu u + \sigma^\alpha |u|^\alpha \beta (\text{sign} u) \tan \frac{\pi \alpha}{2}.
\]

By regression \( y_k = \log \left( -\log |\phi_X(u_k)|^2 \right) \) and \( w_k = \log |u_k| \) in the model

\[
y_k = aw_k + b + \epsilon_k
\]
propose $u_k = \frac{\pi k}{25}, k = 1, 2, \cdots, K$, then

$$\hat{\alpha} = a, \quad \hat{\sigma} = \left(\frac{1}{2} e^b\right)^{\frac{1}{a}}.$$  

By regression $y_l = \left(\arctan \frac{\text{Im}(\phi x(u_l)))}{\text{Re}(\phi x(u_l)))}\right) \frac{1}{u_l}$ and $w_l = \frac{|u_l|^\alpha}{u_l} \text{sign} u_l$ in the model

$$y_l = aw_l + b + \epsilon_l$$

propose $u_l = \frac{\pi l}{50}, l = 1, 2, \cdots, L$, then

$$\hat{\mu} = b, \quad \hat{\beta} = \frac{a}{\hat{\sigma} \hat{\alpha} \tan \frac{\pi \hat{\alpha}}{2}}.$$
**Definition**

The maximum likelihood estimation for $\alpha$-stable distribution based on a vector of observations $x = (x_1, x_2, \cdots, x_n)$, the ML estimate of the parameter vector $\theta = (\alpha, \sigma, \beta, \mu)$ is obtained by maximizing the log-likelihood function

$$L_\theta (x) = \sum_{i=1}^{n} \log \tilde{f} (x_i; \theta)$$

where $\tilde{f} (\cdot; \theta)$ is the stable density function.

- The tild denotes the fact that the explicit form of the stable p.d.f. and have to approximate it numerically.
- The approximated density function can be obtained from inverse transformation of characteristic function through *fast Fourier transformation (FFT)*.
- ML estimates are almost always the most accurate, closely followed by regression-type estimate, quantile method and method of moments.
Werón’s Method

**Theorem**

\[
\epsilon(\alpha) = \text{sign} (1 - \alpha), \quad \gamma_0 = -\frac{\pi}{2} \beta_2 \frac{K(\alpha)}{\alpha},
\]

\[
C(\alpha, \beta_2) = 1 - \frac{1}{4} \left(1 + \beta \frac{K(\alpha)}{\alpha}\right) (1 + \epsilon(\alpha)),
\]

\[
U_\alpha(\gamma, \gamma_0) = \left(\frac{\sin \alpha (\gamma - \gamma_0)}{\cos \gamma}\right)^{\frac{\alpha}{1-\alpha}} \frac{\cos (\gamma - \alpha (\gamma - \gamma_0))}{\cos \gamma}
\]

\[
U_1(\gamma, \beta_2) = \frac{\pi}{2} + \beta_2 \gamma \exp \left(\frac{1}{\beta_2} \left(\frac{\pi}{2} + \beta_2 \gamma\right) \tan \gamma\right),
\]

Then the cumulative function of a standard stable distribution can be written as:

\[
F(x, \alpha, \beta_2) = C(\alpha, \beta_2) + \frac{\epsilon(\alpha)}{\pi} \int_{\gamma_0}^{\pi/2} \exp \left(-x^{\frac{1}{1-\alpha}} U_\alpha(\gamma, \gamma_0)\right) d\gamma, \quad \alpha \neq 1, \ x > 0
\]

\[
F(x, 1, \beta_2) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \exp \left(- \exp \left(-\frac{x}{\beta_2}\right) U_1(\gamma, \beta_2)\right) d\gamma, \quad \alpha = 1, \ \beta_2 > 0.
\]
Theorem

Let $\gamma_0$ be defined as above, $\gamma$ be uniformly distributed on $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ and $W$ be an independent exponential random variable with mean $1$. Then

$$X = \frac{\sin \alpha (\gamma - \gamma_0)}{(\cos \gamma)^{\frac{1}{\alpha}} \left( \cos \left( \frac{\gamma - \alpha (\gamma - \gamma_0)}{W} \right) \right)^{\frac{1-\alpha}{\alpha}}}$$

is $S_\alpha (1, \beta_2, 0)$ for $\alpha \neq 1$.

$$X = \left( \frac{\pi}{2} + \beta_2 \gamma \right) \tan \gamma - \beta_2 \log \left( \frac{W \cos \gamma}{\frac{\pi}{2} + \beta_2 \gamma} \right)$$

is $S_1 (1, \beta_2, 0)$.

More specific proof can be found in Proof of Weron’s Theorem.
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Algorithm for Simulation Random Variable

- Generate a random variable $U$ uniformly distributed on $\left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ and an independent exponential random variable $E$ with mean 1.
- For $\alpha \neq 1$, compute
  \[
  X = S_{\alpha, \beta} \frac{\sin(\alpha (U + B_{\alpha, \beta}))}{(\cos(U))^{\frac{1}{\alpha}}} \left( \frac{\cos(U - \alpha (U + B_{\alpha, \beta}))}{E} \right)^{\frac{1 - \alpha}{\alpha}},
  \]
  where $B_{\alpha, \beta} = \frac{\arctan(\beta \tan(\frac{\pi \alpha}{2}))}{\alpha}$, and $S_{\alpha, \beta} = (1 + \beta^2 \tan^2(\frac{\pi \alpha}{2}))^{\frac{1}{2\alpha}}$.
- For $\alpha = 1$, compute
  \[
  X = \frac{2}{\pi} \left[ \left( \frac{\pi}{2} + \beta U \right) \tan U - \beta \log\left( \frac{\pi}{2} E \cos U \right) \right].
  \]
- Generalize scale and location $Y = \begin{cases} 
\sigma X + \mu, & \alpha \neq 1 \\
\sigma X + \frac{2}{\pi} \beta \sigma \log \sigma + \mu, & \alpha = 1
\end{cases}$.
Algorithms

- **FronTier++/MonteCarlo**: MonteCarlo.cpp, MonteCarlo.h, MCsub.cpp
  - **input file**: domain of simulation, the name of random variable (related parameters), number of simulations, whether simulate stock path (if so, related parameters, such as growth rate, volatility, initial stock price), number of simulation for stock paths
  - **output file**: run-output, approximated cumulative distribution function of the input random variable, specific number of simulated stock paths, average stock path

- **FronTier++/StableParameterEstimation.cpp**
  - **input file**: market data written in a specific file
  - **output**: estimated value of four stable parameters of given data
Numerical Results

With input file:
Domain limit in 0-th dimension: -10 10
Computational grid: 201
Lower bound in 0-th dimension: DIRICHLET_BOUNDARY
Upper bound in 0-th dimension: DIRICHLET_BOUNDARY
Enter name of distribution function: STABLE
Enter alpha: 1.8
Enter beta: 0.0
Enter sigma: 1.0
Enter mu: 0.0
Enter number of samples: 100000
Enter type of random seed: FIXED_SEED
Enter yes for stock simulation: yes
Enter market growth rate: 0.05
Enter market volatility: 0.1
Enter stock start price: 1.0
Enter ending time: 5.0
Enter number of time steps: 1825
Enter number of simulations: 100000
Enter yes to print details: yes
Enter number of print cases: 10
Numerical Results

Numerical Results

X Graph

Y

0.5000
1.0000
1.5000
2.0000
2.5000
3.0000
3.5000
4.0000
4.5000
5.0000
5.5000
6.0000
6.5000

0.0000
1.0000
2.0000
3.0000
4.0000
5.0000
6.0000
7.0000
8.0000
9.0000
10.0000

X Graph

Y

0.0000
0.5000
1.0000
1.5000
2.0000
2.5000
3.0000
3.5000
4.0000
4.5000
5.0000
5.5000
6.0000
6.5000
7.0000
8.0000
9.0000
10.0000

X Graph

Y

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X Graph

Y

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X Graph

Y

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10.0000

X Graph

Y
Numerical Results ($\alpha = 1.5, \beta = -0.8, \sigma = 1.0, \mu = 0.0$)
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**Definition**

Given $\alpha \in (0, 1) \cup (1, 2)$, $C$, $\lambda_+$, $\lambda_- > 0$ and $m \in \mathbb{R}$, random variable $X$ follows *classical tempered stable distribution* (CTS) if the characteristic function is

$$
\phi_X (u) = \phi_{CTS} (u; \alpha, C, \lambda_+, \lambda_-, m)
= \exp \left( ium - iu C \Gamma (1 - \alpha) (\lambda_+^{\alpha-1} - \lambda_-^{\alpha-1}) \right. \\
+ C \Gamma (-\alpha) \left( (\lambda_+ - iu)^{\alpha} - \lambda_+^{\alpha} + (\lambda_- + iu)^{\alpha} - \lambda_-^{\alpha} \right)
$$

and denote $X \sim CTS (\alpha, C, \lambda_+, \lambda_-, m)$.

- The cumulants of $X$ are
  $$
c_1 (X) = m, \quad c_n (X) = C \Gamma (n - \alpha) \left( \lambda_+^{n-\alpha} + (-1)^n \lambda_+^{\alpha-n} \right), \quad n = 2, 3, \cdots.
$$
Tempered Stable Distribution (Cont.)
Definition of Lévy Process

Definition

A cadlag (right-continuity and left limits) stochastic process \((X_t)_{t \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) with value in \(\mathbb{R}\) such that \(X_0 = 0\) is called a Lévy Process if it possesses following properties:

- **Independent increments**: for every increasing sequence of times \(t_0, \cdots, t_n\), the random variables \(X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}\) are independent.
- **Stationary increments**: \(X_{t+h} - X_t\) does not depend on \(t\).
- **Stochastic continuity**: \(\forall \epsilon > 0, \lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0\).
Definition

Given a Lévy process \((X_t)_{t \geq 0}\), assume the stock price is defined by
\[ S_t = S_0 e^{rt+X_t} \]
at every time \(t\) and \(S_0\) is the initial value of the stock price, then the stock price follows an exponential Lévy model. The process \((S_t)_{t \geq 0}\) is referred to as the stock price process and \((X_t)_{t \geq 0}\) is referred as the driving process.

- If the driving process is tempered stable process, then the exponential Lévy model is referred as exponential tempered stable model.
- If the trajectories of \(X\) are neither almost surely increasing nor almost surely decreasing, then the exponential Lévy model given by \(S_t = e^{rt+X_t}\) is arbitrage-free.
Partial Integro-Differential Equation (PIDE)

The value of a European option $C(t, S_t)$ solves a second order PIDE

$$\frac{\partial C}{\partial t}(t, S) + rS \frac{\partial C}{\partial S}(t, S) + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}(t, S) - rC(t, S)$$

$$+ \int \nu(dy) \left[ C(t, Se^y) - C(t, S) - S(e^y - 1) \frac{\partial C}{\partial S}(t, S) \right] = 0$$

Assume $x = \ln S$ and $f(t, x) = C(t, S)$, then the original PIDE formula can be transformed as

$$\frac{\partial f}{\partial t}(t, x) + r \frac{\partial f}{\partial x}(t, x) + \frac{\sigma^2}{2} \left( \frac{\partial^2 f}{\partial x^2}(t, x) - \frac{\partial f}{\partial x}(t, x) \right) - rf(t, x)$$

$$+ \int \nu(dy) \left[ f(t, x + y) - f(t, x) - (e^y - 1) \frac{\partial f}{\partial x}(t, x) \right] = 0$$
References

- Rafel Weron, “Correction to: On the Chambers-Mallows-Stuck method for simulating skewed stable random variables”