1 All-Pairs Shortest Paths (with negative weight edge)

1.1 Johnson’s Algorithm

Johnson’s algorithm consists of the following steps:

- \( O(|V|) \), a new node \( q \) is added to the graph and connected by zero-weight edges to each of the other nodes, \( G' = G \cup \{q\}, \{q \rightarrow v \mid \forall v \in G\} \).

- \( O(|E||V|) \), compute for each vertex \( v \) the minimum weight \( h(v) \) of a path from \( q \) to \( v \), i.e. \( h[v] = d(q, v), \forall v \) (Bellman-Ford). If this step detects a negative cycle, the algorithm is terminated.

- \( O(|E|) \), the edges of the original graph are reweighted using the values computed in last step and let \( w'(u, v) = w(u, v) + h[u] - h[v] \).

- \( O \left( |V||E| + |V|^2 \log |V| \right) \), \( q \) is removed and Dijkstra’s algorithm is used to find the shortest paths from each node to every other vertex in the

Figure 1.1: Example of Johnson’s Algorithm
rewighted graph. For each $u \in V$, compute $d'[u][v] = d'(u, v)$ in $(G, w')$ (Dijkstra’s).

- $O\left(|V|^2\right)$, the reweighted shortest path is transformed to the original form.

For each $u, v \in V \times V$, compute $d[u][v] = d'(u, v) - h[u] + h[v]$.

The operation time of each step is listed in the beginning of each row.

### 1.2 Further Analysis

**Proposition 1.** $w'(u, v) \geq 0$, $\forall u, v$.

Proof:

$$q \xrightarrow{h[u]} \ast u \xrightarrow{w(u,v)} \ast v, q \xrightarrow{h[v]} \ast v$$

so $h[v] \leq h[u] + w(u, v)$, hence $w'(u, v) = w(u, v) + h[u] - h[v] \geq 0$.

**Proposition 2.** For any path $p : u_1 \rightarrow \ast u_1 \in G$

$$w'(p) = w(p) + h[u_1] - h[u_1]$$

Proof:

$$w'(p) = \sum_{(u, v) \in p} w'(u, v) = \sum_{(u, v) \in p} [w(u, v) + h[u] - h[v]]$$

$$= \sum_{(u, v) \in p} w(u, v) + h[u_1] - h[u_1]$$

**Corollary 3.** $d_w(u, v) = d_w'(u, v) - h[u] + h[v]$  

### 2 Review

Final exam form:

- 5 questions in total
- 1/3 before midterm (2 questions)
- 2/3 after midterm (3 questions)

Summary:

- hashing and universal hashing
- balls and bins (power of two choice)
- linear probing (quotient filter)
- matrix vector multiplication
• divide and conquer

• hash table vs. tree: hash table is applied to solve problem whose keys are not in order; tree is applied to problem with comparison result, such as finding elements between two ordered number.

• median finding $O(n)$

• extra memory (such as epsilon tree)

• unix file system: access time (every access), modify time, creation time.