1 Master Theory

The master theory provides a method for solving recurrences of the form

\[ T(n) = aT\left(\frac{n}{b}\right) + f(n) \]

where \( a \geq 1 \) and \( b \geq 1 \) are constants and \( f(n) \) is an asymptotic positive function.

**Theorem 1.** Let \( a \geq 1 \) and \( b \geq 1 \) be constants, let \( f(n) \) be a function and let \( T(n) \) be defined on the nonnegative integers by the recurrence

\[ T(n) = aT\left(\frac{n}{b}\right) + f(n) \]

where we interpret \( \frac{n}{b} \) to mean either \( \lfloor \frac{n}{b} \rfloor \) or \( \lceil \frac{n}{b} \rceil \). Then \( T(n) \) can be bounded asymptotically as follows:

1. If \( f(n) = O\left(n^{\log_b a - \epsilon}\right) \) for some constant \( \epsilon > 0 \), then \( T(n) = \Theta\left(n^{\log_b a}\right) \).
2. If \( f(n) = \Theta\left(n^{\log_b a} \left(\log n\right)^k\right) \), then \( T(n) = \Theta\left(n^{\log_b a} \left(\log n\right)^{k+1}\right) \), \( k \geq 0 \).
3. If \( f(n) = \Omega\left(n^{\log_b a + \epsilon}\right) \) for some constant \( \epsilon > 0 \), and if \( af\left(\frac{n}{b}\right) \leq cf(n) \) for some constant \( c < 1 \) and sufficiently large \( n \), then \( T(n) = \Theta\left(f(n)\right) \).

There are some examples about how to apply the master theorem.

**Example 2.** \( T(n) = 8T\left(\frac{n}{2}\right) + O\left(n^2\right) \)
In this example, $a = 8$, $b = 2$, then $\log_b a = 3$, $f(n) = O(n^3)$. Thus $T(n) = \Theta(n^3)$.

**Example 3.** $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$

In this example, $a = 2$, $b = 2$, then $\log_b a = 1$, $f(n) = \Theta(n)$. Thus $T(n) = \Theta(n \log n)$.

There are some examples which contradict some assumptions of master theory.

**Example 4.** $T(n) = 2^n T\left(\frac{n}{2}\right) + O(n)$

In this example, $a = 2^2$ is not a constant.

**Example 5.** $T(n) = T(\sqrt{n}) + O(n)$

In this example, the $n$ inside $T(\cdot)$ is not linear.

**Example 6.** $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

In this example, $f(n) = \frac{n}{\log n}$ and according to the second case $k = -1 < 0$. Thus, we cannot use master theory.

**Example 7.** $T(n) = T\left(\frac{n}{2}\right) + n \cos n$

In this example, $a = 1$, $b = 2$, $f(n) = n \cos n$ and $n^{\log_b a} = 1$. Since $f(n) = \Omega(n^\epsilon)$ for some constant $\epsilon > 0$, $af\left(\frac{n}{2}\right) = \frac{n}{2} \cos \frac{n}{2}$ and $cf(n) = cn \cos n$, thus $af\left(\frac{n}{2}\right) \leq cf(n)$ does not hold for some constant $c < 1$ and sufficiently large $n$. The master theory cannot be applied in this example.
2 The Closest Pair Problem

Problem 8. Given a set of $n$ points $\{p_1, \cdots, p_n\} = \{(x_1, y_1), \cdots, (x_n, y_n)\}$, what is the closest pair of points?

Definition 9. Define Euclidean distance

$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}.$$ 

The naive algorithm can be summarized as:

- For all pairs $p_i$ and $p_j$, find the distance between $p_i$ and $p_j$
- Find the minimum distance

This algorithm takes $O(n^2)$. And the lower bound of this closest pair problem takes $\Omega(n \log n)$ which comes from duplicate element algorithm. If there are two same elements, then the distance of closest pair is 0. Thus, this two algorithms are somehow similar in essence.

A high-level approach is:

$$\text{ClosestPair (all points)}$$

if $n = 2$

return distance

else

$\text{ClosestPair (all points in right side)}$
$\text{ClosestPair (all points in left side)}$

$\delta = \min \{\text{left distance, right distance}\}$
$\text{ClosestSplitPair (all points in right sides, all points in left side, } \delta)$

return $\min \{\delta, \text{split distance}\}$

And the algorithm for finding closest pair in split parts is:
ClosestSplitPair (all points in right sides, all points in left side, \(\delta\))

\(\bar{x}\) = largest \(x\) - coordinate in left side

\(S_y\) = points with \(x\)-coordinate in \([\bar{x} - \delta, \bar{x} + \delta]\) and sorted by \(y\)-coordinate

best distance = \(\delta\)

best pair = NULL

for \(i = 1\) to \(|S_y| - 7\)

for \(j = 1\) to 7

\(p\) = \(i\)th point of \(S_y\)

\(q\) = \((i + j)\)th point of \(S_y\)

if \(d(p, q) < \text{best distance}\)

best pair = \((p, q)\)

best distance = \(d(p, q)\)

This finding closest pair in split parts algorithm takes \(O(n)\) operations and assume \(T(n)\) be the operation time of finding closest pair algorithm

\[T(n) = 2T\left(\frac{n}{2}\right) + O(n)\]

then based on the master theory \(T(n) = O(n \log n)\) is the operation time.

**Theorem 10.** Let \(p, q\) be a split pair with \(d(p, q) < \delta\), then

- \(p\) and \(q\) are members of \(S_y\)
- \(p\) and \(q\) are at most 7 positions apart in \(S_y\)

Proof:

Given \(p = (x_1, y_1), q = (x_2, y_2)\) are from left side and right side correspondingly, since \(d(p, q) < \delta\), we have \(|x_1 - x_2| < \delta, |y_1 - y_2| < \delta\). With \(x_1 \leq \bar{x} < x_2\), there are

\[|\bar{x} - x_1| < \delta, |\bar{x} - x_2| < \delta \Rightarrow x_1, x_2 \in S_y.\]

The second part can be proved with following Lemma. \(\square\)
Lemma 11. Draw eight $\frac{\delta}{2} \times \frac{\delta}{2}$ boxes with center $\bar{x}$ and bottom line $\min \{y_1, y_2\}$. All points in $S_y$ with $y$-coordinate between those of $p$ and $q$, inclusive, lies in one of these 8 boxes. And there is at most one point in each box.

Proof:

First, recall $y$-coordinates of $p, q$ differ by $< \delta$. And by definition of $S_y$, all $x$-coordinates between $\bar{x} - \delta$ and $\bar{x} + \delta$. Thus $p, q$ must be included.

Second, assume there are two points in same box, then $d(p, q) < \frac{\delta}{2}\sqrt{2} < \delta$ which contradicts the conclusion that the minimum distance of both sides is $\delta$. Thus, the conclusion is proved. □