1 Quick Select (Cont.)

First review the randomized $k$th finding algorithm:

\[
\text{\textit{kth}}(A[0, \cdots, n-1], k) \\
\text{if } n \leq 1 \\
\text{\hspace{1cm} \textbf{return} } A[0] \\
\text{else} \\
\text{\hspace{1cm} } i \leftarrow \text{random}(0, 1, \cdots, n-1) \\
\text{\hspace{1cm} } \text{\textit{swap}}(A[i], A[n-1]) \\
\text{\hspace{1cm} } t = \text{\textit{partition}}(A[0, \cdots, n-2], A[n-1]) \\
\text{\hspace{1cm} } \text{\textit{swap}}(A[t], A[n-1]) \\
\text{\hspace{1cm} if } t > k \\
\text{\hspace{2cm} \textit{kth}}(A[0, \cdots, t-1], k) \\
\text{\hspace{1cm} else} \\
\text{\hspace{2cm} \textit{kth}}(A[t, \cdots, n-1], k-t)
\]

To estimate running time of the algorithm

\[
T(n, k) = \text{average running time of } \textit{kth}(A, k)
\]

\[
T(n) = \max_k T(n, k)
\]

then, based on the algorithm

\[
T(n, k) = \sum_{t=1}^{k-1} \frac{1}{n-t} T(n-t, k-t) + \sum_{t=k}^{n-1} \frac{1}{n-t} T(t, k) + O(n)
\]

\[
\leq \sum_{t=1}^{k-1} \frac{1}{n-t} T(n-t) + \sum_{t=k}^{n-1} \frac{1}{n-t} T(t) + O(n)
\]

\[
= \sum_{t=n-k+1}^{n} \frac{1}{n-t} T(t) + \sum_{t=k}^{n-1} \frac{1}{n-t} T(t) + O(n)
\]
\[
T(n) \leq C'n
\]

then

\[
C'n \leq 2 \sum_{t=1}^{n-1} \frac{1}{n-1} C't + Cn
\]

\[= \frac{2C'}{n-1} \cdot \frac{n(n-1)}{2} + Cn = C'n + Cn
\]

2 Median of Median Algorithm

The implement of median of median algorithm can be divided into following steps:

- divide original array of length \(n\), into \(\frac{n}{5}\) subarrays with length 5
- find the median number of each subarray \(m_1, \ldots, m_{\frac{n}{5}}\)
- find the median number of \(\{m_1, \ldots, m_{\frac{n}{5}}\}\) and use it as the pivot to do later select work

The algorithm is as followed:

\[
kth(A[0, \ldots, n-1], k)
\]

if \(n \leq 5\)

- do base case
else

(1) for \(i = 0\) to \(\frac{n-1}{5}\)
(2) sort \(A[5i, \ldots, 5(i+1) - 1]\)
(3) \(t = kth(A[2, 7, 12, \ldots, n-3], \frac{n}{10})\)
(4) \(i = partition(A, t)\)
(4) if \(i > k\)
(4) \(kth(A[0, \ldots, i-1], k)\)
(4) else
(4) \(kth(A[i, \ldots, n-1], k-i)\)

Among \(\frac{n}{5}\) medians of corresponding subarrays, the chosen pivot is larger than \(\frac{7n}{10}\) medians. Since these medians come from subarrays with length 5, the chosen pivot is larger than at least \(\frac{3n}{10}\) elements of original array. Similarly, we can prove that the chosen pivot is smaller than at least \(\frac{3n}{10}\) elements of original array. In sum, \(\frac{3n}{10} \leq rank\ of\ pivot \leq \frac{7n}{10}\.

Among all these operation steps:
• step (1) takes $O(n)$
• step (2) takes $T\left(\frac{n}{5}\right)$
• step (3) takes $O(n)$
• since $\frac{3n}{10} \leq \text{rank of pivot} \leq \frac{7n}{10}$, step (4) takes $T\left(\frac{7n}{10}\right)$

Summarize above results

$$T(n) \leq T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + O(n)$$

$$\leq T\left(\frac{7n}{10}\right) + T\left(\frac{n}{5}\right) + Cn$$

Guess that $T(n) \leq C'n$

$$C'\frac{7n}{10} + C'\frac{n}{5} + Cn \leq C'n$$

$$\Rightarrow C'\frac{9n}{10} + Cn \leq C'n$$

$$\Rightarrow C' \geq 10C, \ T(n) = O(n)$$