AMS261 Recitation Course 13

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Outline

1. Review
   - Vector Fields
   - Line Integrals
   - Conservative Vector Fields and Independence of Path

2. Recitation Problems
Definition of Vector Field

- A vector field over a plane region $R$ is a function $\vec{F}$ that assigns a vector $\vec{F}(x, y)$ to each point in $R$.

- A vector field over a solid region $Q$ in space is a function $\vec{F}$ that assigns a vector $\vec{F}(x, y, z)$ to each point in $Q$.

- A vector field $\vec{F}(x, y, z) = M(x, y, z)\vec{i} + N(x, y, z)\vec{j} + P(x, y, z)\vec{k}$ is continuous at a point if and only if each of its component functions $M$, $N$ and $P$ is continuous at that point.

- Some physical examples of vector fields are velocity fields, gravitational fields and electric force fields.
Conservative Vector Fields

Definition of Conservative Vector Field

A vector field $\vec{F}$ is called conservative if there exists a differentiable function $f$ such that $\vec{F} = \nabla f$. The function $f$ is called the potential function for $\vec{F}$.

Test for Conservative Vector Field in the Plane

Let $M$ and $N$ have continuous first partial derivatives on an open disk $R$. The vector field given by $\vec{F}(x, y) = M\vec{i} + N\vec{j}$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
Curl of a Vector Field

Definition of Curl of a Vector Field

The curl of $\vec{F}(x, y, z) = M\vec{i} + N\vec{j} + P\vec{k}$ is

$$\text{curl } \vec{F}(x, y, z) = \nabla \times \vec{F}(x, y, z)$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}.$$ 

Test for Conservative Vector Field in Space

Suppose that $M$, $N$ and $P$ have continuous first partial derivatives in an open sphere $Q$ in space. The vector field given by $\vec{F}(x, y) = M\vec{i} + N\vec{j} + P\vec{k}$ is conservative if and only if

$$\text{curl } \vec{F}(x, y, z) = \vec{0}.$$ 

That is, $\vec{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$
Definition of Divergence of a Vector Field

The divergence of \( \vec{F}(x, y) = M\vec{i} + N\vec{j} \) is

\[
\text{div } \vec{F}(x, y) = \nabla \cdot \vec{F}(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.
\]

The divergence of \( \vec{F}(x, y) = M\vec{i} + N\vec{j} + P\vec{k} \) is

\[
\text{div } \vec{F}(x, y, z) = \nabla \cdot \vec{F}(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.
\]

If \( \text{div } \vec{F} = 0 \), then \( \vec{F} \) is said to be divergent free.
### Piecewise Smooth Curves

#### Definition of Smooth and Piecewise Smooth

- A plane curve $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \ a \leq t \leq b$ is smooth if
  
  $$\frac{dx}{dt}, \frac{dy}{dt}$$

  are continuous on $[a, b]$ and not simultaneously 0 on $(a, b)$.

- A space curve $C : \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}, \ a \leq t \leq b$ is smooth if
  
  $$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$$

  are continuous on $[a, b]$ and not simultaneously 0 on $(a, b)$.

- A curve $C$ is piecewise smooth if the interval $[a, b]$ can be partitioned into a finite number of subintervals, on each of which $C$ is smooth.
Line Integrals

Definition of Line Integrals

If $f$ is defined in a region containing a smooth curve $C$ of finite length, then the line integral of $f$ along $C$ is given by

$$\int_C f(x, y) \, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta s_i$$

or

$$\int_C f(x, y, z) \, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta s_i$$

provided this limit exists.
Evaluation of a Line Integral as a Definite Integral

Let \( f \) be continuous in a region containing a smooth curve \( C \). If \( C \) is given by \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \), where \( a \leq t \leq b \) then

\[
\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt.
\]

If \( C \) is given by \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \), where \( a \leq t \leq b \) then

\[
\int_C f(x, y, z) \, ds
= \int_a^b f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2} \, dt.
\]
Line Integrals of Vector Fields

Definition of the Line Integral of a Vector Field

Let $\vec{F}$ be a continuous vector field defined on a smooth curve $C$ given by $\vec{r}(t)$, where $a \leq t \leq b$. The line integral of $\vec{F}$ on $C$ is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} \, ds = \int_C \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t) \, dt.$$ 

$$= \int_C \left( M\vec{i} + N\vec{j} \right) \cdot \left( x'(t)\vec{i} + y'(t)\vec{j} \right) \, dt$$

$$= \int_C \left( M \frac{dx}{dt} + N \frac{dy}{dt} \right) \, dt = \int_C M \, dx + N \, dy$$

- This differential form can be extended to three variables as

$$\int_C M \, dx + N \, dy + P \, dz$$
Fundamental Theorem of Line Integrals

Definition of the Line Integral of a Vector Field

Let $C$ be a piecewise smooth curve lying in an open region $R$ and given by

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad a \leq t \leq b.$$  

If $\vec{F}(x, y) = Mi + Nj$ is conservative in $R$, and $M$ and $N$ are continuous in $R$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where $f$ is a potential function of $\vec{F}$. That is $\vec{F}(x, y) = \nabla f(x, y)$. 

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Independence of Path

Equivalent Conditions

Let $\vec{F}(x, y, z) = M\vec{i} + N\vec{j} + P\vec{k}$ have continuous first partial derivatives in an open connected region $R$, and let $C$ be a piecewise smooth curve in $R$. The following conditions are equivalent.

- $\vec{F}$ is conservative. That is $\vec{F} = \nabla f$ for some function $f$.
- $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.
- $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve $C$ in $R$. 

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15.1.81 on p.1068

Find $\text{div}(\text{curl } \vec{F}) = \nabla \cdot \left( \nabla \times \vec{F} \right)$ where $\vec{F}(x, y, z) = xyz \hat{i} + y \hat{j} + z \hat{k}$.

Solution:

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = 0 \cdot \hat{i} - xy \hat{j} + xz \hat{k}$$

$$\text{div} \left( \text{curl } \vec{F} \right) = \text{div} \left( 0 \cdot \hat{i} - xy \hat{j} + xz \hat{k} \right) = 0 - x + x = 0$$
15.2.73 on p.1082

Find the moments of inertia for the wire of density $\rho$. A wire lies along $\vec{r}(t) = a \cos t \hat{i} + a \sin t \hat{j}$, $0 \leq t \leq 2\pi$ and $a > 0$, with density function $\rho(x, y) = 1$.

Solution:
- Since $\vec{r}'(t) = -a \sin t \hat{i} + a \cos t \hat{j}$
  \[ \sqrt{[x'(t)]^2 + [y'(t)]^2} = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a \]
- The moments of inertia are
  \[
  I_x = \int_C y^2 \rho(x, y) \, ds = \int_0^{2\pi} y^2 \rho(x, y) \sqrt{[x'(t)]^2 + [y'(t)]^2} \, dt \\
  = \int_0^{2\pi} (a \sin t)^2 \cdot a \, dt = a^3 \int_0^{2\pi} \sin^2 t dt = a^3 \pi \\
  I_y = \int_C x^2 \rho(x, y) \, ds = a^3 \int_0^{2\pi} \cos^2 t dt = a^3 \pi \]
Find the work done by the force field \( \vec{F}(x, y) = 9x^2y^2 \hat{i} + (6x^3y - 1) \hat{j} \) in moving an object from \( P(0, 0) \) to \( Q(5, 9) \).

Solution:

- Since \( \frac{\partial(9x^2y^2)}{\partial y} = 18x^2y = \frac{\partial(6x^3y-1)}{\partial x} \), \( \vec{F}(x, y) \) is conservative.
- The work done by the force field is

\[
\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(x(b), y(b)) - f(x(a), y(a))
\]

\[
= (3x^3y^2 - y) \bigg|_{(0,0)}^{(5,9)} = 30366.
\]
15.4.31 on p.1100

Use Green’s theorem to verify the line integral formulas. The centroid of the region have area $A$ bounded by the simple closed path $C$ is

$$
\bar{x} = \frac{1}{2A} \int_C x^2 \, dy \quad \text{and} \quad \bar{y} = -\frac{1}{2A} \int_C y^2 \, dx.
$$

Proof:

- Based on Green’s formula

$$
\int_C M \, dx + N \, dy = \int_R \int \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
$$

$$
\frac{1}{2A} \int_C x^2 \, dy = \frac{1}{2A} \int_R \int \left( \int_C x \, dA \right) = \bar{x}
$$

$$
- \frac{1}{2A} \int_C y^2 \, dx = -\frac{1}{2A} \int_R \int \left( -2y \right) \, dA = \bar{y}
$$

Conclusion is proved.
Find the mass of the surface lamina $S : 2x + 3y + 6z = 12$ first octant with density $\rho(x, y, z) = x^2 + y^2$.

Solution:

- The projection of $S$ in first octant on $xy$-plane is $x \geq 0, y \geq 0, 2x + 3y \leq 12$, then the integration domain is

$$0 \leq x \leq 6, \ 0 \leq y \leq 4 - \frac{2}{3}x$$
The mass of the surface $z = f(x, y) = 2 - \frac{1}{3}x - \frac{1}{2}y$ is

\[
m = \int \int \rho ds = \int \int \rho \sqrt{1 + f_x^2 + f_y^2} dA
\]

\[
= \int \int (x^2 + y^2) \sqrt{1 + \frac{1}{9} + \frac{1}{4}} dA
\]

\[
= \frac{7}{6} \int_{0}^{6} \int_{0}^{4 - \frac{2}{3}x} (x^2 + y^2) dydx
\]

\[
= \left[ \frac{4}{3}x^2 - \frac{1}{6}x^4 - \frac{1}{8} \left( 4 - \frac{2}{3}x \right)^4 \right]_0^6 = \frac{364}{3}.
\]