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System of wave equations

\[ \nabla_t + A \nabla_x = 0 \]

\[ \nabla = [v_1, v_2, \ldots, v_k]^T \]

primitive variables.

A diagonalizable constant

e.g.

\[ v_{1t} + v_{1x} + 4v_{2x} = 0 \]

\[ v_{2t} + 4v_{1x} + v_{2x} = 0 \]

\( k=2 \)

\( v_1, v_2 \) primitive variable.

\[ A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \]

\[ \nabla = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \]

\[ \nabla_t + A \nabla_x = 0 \]

eigenvalues of \( A \).

\[ \det (A I - A) = 0 \]
\[ \lambda_1 = 5, \quad \lambda_2 = -3 \]

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}}
\end{bmatrix}
\text{orthonormal}
\]

\[
P = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}
\quad D = \begin{bmatrix}
5 & 0 \\
0 & -3
\end{bmatrix}
\]

\[P^{-1}AP = D.\]

\[P^{-1} = P^T = \begin{bmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{bmatrix}\]

\[A \text{ is } k \times k\]

\[
\begin{bmatrix}
P & I \\
I & P^{-1}
\end{bmatrix}
\]

\[
\vec{V}_t + A \vec{V}_x = 0 \quad D = P^{-1}AP
\]

\[
P^{-1} \vec{V}_t + P^{-1}AP \vec{V}_x = 0
\]

\[
P^{-1} \vec{V}_t + [\boxed{P^{-1}AP}] P^{-1} \vec{V}_x = 0
\]
\[
P^{-1} \vec{V}_t + D P^{-1} \vec{V}_x = 0
\]

let \( \vec{W} = P^{-1} \vec{V} \)

\[
\vec{W}_t + D \vec{W}_x = 0
\]

decoupled system is

\[
\omega_i t + \lambda_i \omega_i x = 0
\]

\( \omega \) characteristic variables, (or Riemann invariants)

\( \vec{V} \) primitive variables,

\[
\vec{V}_t + A \vec{V}_x = 0
\]

place a grid \( \Delta X \).

functions defined at \( n \)-th time step

\[
U^n_k = [u_{ik}^n, u_{2k}^n, \ldots, u_{nk}^n]^T
\]

first \( k \) variable index

second \( k \), mesh index.
\[ \mathbf{u}^n = [ \ldots, \mathbf{u}_{-1}, \mathbf{u}_0, \mathbf{u}_1, \ldots ] \]

Mesh index

general two level difference scheme
\[ \mathbf{u}^{n+1} = Q \mathbf{u}^n \]

Take D.F.T.
\[ \hat{\mathbf{u}}^{n+1} = G(\tau) \hat{\mathbf{u}}^n \]

\[ G(\tau) \text{ symbol or amplification matrix} \]

Scalar.
\[ \hat{\mathbf{u}}^{n+1} = P(\tau) \hat{\mathbf{u}}^n \]

\[ |P(\tau)| \leq 1 \quad \text{stable} \]

Theorem 6.2.1.

The difference scheme is stable with respect to L2 norm if and only if there exist positive constants \( \Delta \), \( \Delta_0 \), \( \kappa \), and \( \beta \) so that
\[ \| G^n(\tau) \|_2 \leq \kappa e^{\beta t} \]
for $0 < \alpha \leq \alpha_0$, $0 < c = c_0$, $t = n \tau$ and all $\varepsilon \in [-\pi, \pi]$. 

Theorem 6.1.2.

If the difference scheme $u^n = Q u^n$ is stable in the $L_2$, $\alpha x$ norm, then there exists positive constant $\alpha_0$, $c_0$, and $K$, independent of $\alpha$, $\alpha x$, and $\varepsilon$, so that

$$\Gamma(G(\varepsilon)) \leq (1 + c \varepsilon)$$

for $0 < c = c_0$, $0 < \alpha \leq \alpha_0$, $\varepsilon \in [-\pi, \pi]$.

$\Gamma(G(\varepsilon))$ spectral radius of the amplification matrix $G$.

Scalar equation $|P(\varepsilon)|$

Vector equation $\Gamma(G(\varepsilon))$

If matrix $G$ is normal ($G^* G = G G^*$)

$$\Gamma(G) = \|G\|_2$$

$G^*$ conjugate transpose.
Theorem 6.2.3.
Suppose $G$ is normal ($G^*G = GG^*$), and there exist positive constant $\delta k_0, \delta t_0$ and $C$, independent of $\delta V, \delta t$ and $\delta$, so that $\nabla (G(\beta)) \leq 1 + C \delta t$
for $0 < \alpha < \alpha_0$, $0 < \delta t < \delta t_0$, $\beta \in [-\pi, \pi]$ then difference scheme is stable.

$\nabla (G(\beta)) \leq 1 + C \delta t$

— Von Neumann Condition

If normal
$\nabla (G(\beta)) \leq 1 + C \delta t$

then $\Rightarrow$ stable.

Theorem 6.2.4.
$\bar{u}^n = Q \bar{u}^n$

Suppose $G$ satisfies the Von-Neumann condition.

Then
1) if $Q$ is self adjoint (Hermitian) then the scheme is stable.
(2) If there exists an operator $S$ such that $\|S\|_{L^2,\mathcal{A}} \leq C_1$, $\|S^{-1}\|_{L^2,\mathcal{A}} \leq C_2$, (with $C_1, C_2$ constant) and $SGS^{-1}$ is self-adjoint, then the scheme is stable.

(3) If $G$ is Hermitian, the scheme is stable.

(4) If there exists a matrix $S$ such that $\|S\|_{L^2,\mathcal{A}} \leq C_1$, $\|S^{-1}\|_{L^2,\mathcal{A}} \leq C_2$, and $SGS^{-1}$ is Hermitian, the scheme is stable.

(5) If the elements of $G(\mathcal{A})$ are bounded and if all the eigenvalues of $G$, with the possible exception of one, lie, in a circle inside the unit circle, the scheme is stable.
Theorem 6.2.5.

1. If there exists a non-negative C such that \( G^*G \leq (1+\text{Cot})I \), then the scheme is stable.

2. If there exist a matrix S such that \( \|S\|_2 \leq C_1 \), \( \|S^*\|_2 \leq C_2 \), and non-negative C such that \( H = SGS^* \) satisfies \( H^*H \leq (1+\text{Cot})I \), then the scheme is stable.

Numerical scheme

1. Forward differencing

\[
\begin{align*}
\frac{\mathbf{u}_{k+1} - \mathbf{u}_k}{\Delta t} &= \mathbf{u}_k - \frac{\Delta t}{\Delta x} A \left( \mathbf{u}_{k+1} - \mathbf{u}_k \right) \\
\mathbf{u}_0 &= \text{vector of vectors} \\
\mathbf{u}_{k+1} &= -\frac{\Delta t}{\Delta x} A \mathbf{u}_{k+1} + \left( I + \frac{\Delta t}{\Delta x} A \right) \mathbf{u}_k
\end{align*}
\]

Take O.F.T. \( R = \frac{\Delta t}{\Delta x} \).
\[ \hat{u}^{n+1}(\bar{\gamma}) = -Re^{i\delta} \hat{u}^{n}(\bar{\gamma}) + (I+RA)\hat{u}^{n}(\bar{\gamma}) \]

\[ G(\bar{\gamma}) = -R \cos \theta A - R \sin \theta A + I + RA \]

\[ A \text{ is diagonalizable.} \quad P P^T A P = D \]

\[ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} \quad D = S^{-1} A S \]

\[ H = S^{-1} G S \]

\[ = S^{-1} (-R \cos \theta A - R \sin \theta A + I + RA) S \]

\[ = I - R \cos \theta S^{-1} A S - i R \sin \theta S^{-1} A S + R \theta A \]

\[ = I - R \cos \theta D - i R \sin \theta D + RD \]

\[ H \text{ is diagonal.} \]

\[ \lambda_i (\bar{\gamma}) = (-R \cos \theta \lambda_i + i R \sin \theta \lambda_i) \]

\[ i = 1, \ldots, K \]
\[ \overrightarrow{V}_t + A \overrightarrow{V}_x = 0. \quad \overrightarrow{W} = s^T \overrightarrow{V} \]
\[ \Rightarrow \overrightarrow{W}_t + D \overrightarrow{W}_x = 0. \]
\[ \hat{w}_i + \lambda_i \hat{w}_i = 0. \quad \text{ith wave equation} \]

* The symbol of the forward difference scheme for solving the ith equation is \( \hat{w}(i) \).

\[ |\hat{w}(i)| = |\hat{w}_i| = 0 \Rightarrow \lambda_i \leq 0. \]
\[ |\Delta i_1 \hat{w}_t \overrightarrow{\Delta x}| \leq 1 \]

H diagonal matrix.

\[ H^* H \text{ diagonal} \]
\[ |\hat{w}(i)|^2 \text{ on the diagonal} \]

\[ |\hat{w}(i)| = 1 \Rightarrow H^* H \leq I \]

Theorem 6.2.5. (2)

\[ \Rightarrow \lambda_i \leq 0, \quad \frac{\lambda_i \Delta t}{\Delta x} \leq 1. \quad \text{Stable scheme.} \]