Implicit

7. BTFS
\[ u_{n+1}^k = u^n_k - R \delta^+ u_{n+1}^k \quad R \geq 0 \quad \text{stable} \]

BTBS
\[ u_{n+1}^k = u^n_k - R \delta^- u_{n+1}^k \quad R \geq 0 \quad \text{stable} \]

\[ O(\delta t) + O(\delta x) \]

8. BTCS
Central implicit
\[ u_{n+1}^k = u^n_k - \frac{R}{2} \delta^0 u_{n+1}^k \]

\[ O(\delta t^2) + O(\delta x^2) \]
unconditionally stable

9. Implicit L-W
\[ u_{n+1}^k = u^n_k - \frac{R}{2} \delta^0 u_{n+1}^k + \frac{k^2 \delta^2}{2} u_{n+1}^k \]

\[ O(\delta t^2) + O(\delta x^2) \]
unconditionally stable
\( c - N \)

\[
    u_{i+1}^n = u_i^n + \frac{R}{4} \Delta t \Delta x \left( u_i^n - \frac{R}{4} \Delta t \Delta x u_i^n \right) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)
\]

\[
    |PCg| = 1
\]

**CFL condition**

Da analytic domain of dependence is contained in the numerical domain of dependence \( D_n \).

* Necessary condition for convergence

\[
    u^{n+1}_k = \alpha u^n_k + \beta u^{n-1}_k
\]

\( D_n = [(k-N-1)\Delta x , k\Delta x] \)

\( D_a = [(k-R(m+1))\Delta x , k\Delta x] \).

**CFL condition :** \( 0 \leq R \leq 1 \)
Further generalize it
\[ U_k^{m_1} = \sum_{m_2} U_k^n \]
needs \( U_{k-m_1}, \ldots, U_{k+m_2} \)

\[
D_n = \left[ (k-m_1)(n+1) \right] dx, \quad \left[ k + m_2(n+1) \right] dx
\]

\[
D_a = \left[ k - R(n+1) \right] dx
\]

CTF condition: \[-m_1 \leq R \leq m_2, \quad -m_2 \leq R \leq m_1 \]

no explicit, consistent, and unconditionally
stable scheme for hyperbolic equation.

Numerical dispersion and dissipation

\[
V(x, t) = V e^{i(c(\omega \tau + \beta x)}
\]

\( \omega \) frequency of the wave
\( \beta \) wave number
\[ V_t = \gamma V_{xx} \]
\[ V(x,t) = \hat{V} e^{-\gamma t^2} e^{i\beta x} \]
\[ \omega = +i\beta^2 \quad \text{dispersion relation} \]

Wave doesn't move, but decays with time.

\[ V_t + a V_x = 0. \]
\[ V(x,t) = \hat{V} e^{i\beta(x-ax)} \]
\[ \omega = -\alpha \beta \quad \text{dispersion relation} \]

Wave: no decay of amplitude, moves with a speed dependent of frequency.

Dissipation: decay with time.

Dispersion: different wave number modes propagate at different speeds.
PDE contains only even ordered $x$ derivatives
$\rightarrow$ dissipative.

Contains only odd ordered $x$ derivatives
$\rightarrow$ dispersive if order is greater than 1.

numerical dissipative.

$u_t + a u_x = D_n u_{xxx} + \ldots$

$u_t + a u_x = 2 u_{xxx} + \ldots$

numerical dispersion.

upwind scheme:

$$D_n = \frac{1}{2} a a x \left(-1 \eta^2 \right)$$

L-F scheme:

$$D_n = \frac{1}{2} \frac{a^2 x^2}{\alpha^2} \left(-1 \eta^2 \right)$$

central implicit:

$$D_n = \frac{1}{2} a^2 \alpha t .$$

First order scheme will introduce numerical dissipation.
\[ \lambda = \frac{a}{6} (R^2 - 1) \sigma x^2 \]

\[ \lambda = \frac{a}{6} (R^2 - 1) \sigma x^2 \]

\[ \lambda = -\frac{1}{6} a \sigma x^2 \]

Tridiagonal system of equations:

\[ A \bar{u} = \bar{d} \]

\[ \bar{u} = \begin{pmatrix} u_1 & \cdots & u_{n-1} \\ u_{n-1} \end{pmatrix} \]

\[ \bar{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_{n-1} \end{pmatrix} \]

\[ A = \begin{pmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & \ddots & \vdots \\ 0 & a_3 & b_3 & c_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{N-1} & b_{N-1} & c_{N-1} \end{pmatrix} \]

\[ u_{i+1} = A_i u_i + B_i \quad i = 1, \ldots, N-1 \]

\[ A_{N-1} = -\frac{a_{N-1}}{b_{N-1}} \quad B_{N-1} = \frac{d_{N-1}}{b_{N-1}} \]

\[ A_i = \frac{-a_i}{A_i c_i + b_i} \quad B_i = \frac{d_i - B_i c_i}{A_i c_i + b_i} \quad i = 2, N-1 \]
\[ \Rightarrow \{ A_{N-1}, B_{N-1} \}, \ldots, \{ A_1, B_1 \} \]

\[ u_i = \frac{d_i - B_i c_i}{A_i c_i + b_i} \]

\[ u_{i+1} = A_i c_{i+1} + B_i \quad i = 1, \ldots, N-1 \]

\[ \Rightarrow u_1, \ldots, u_N. \]