(1). Determine the order of accuracy of the following difference equation to the given problem.
(a) Explicit scheme (FTCS)
\[ u_{k}^{n+1} = u_{k}^{n} - \frac{a\Delta t}{2\Delta x} \delta_{0} u_{k}^{n} + \frac{\nu\Delta t}{\Delta x^2} \delta_{2} u_{k}^{n} \]
\[ v_{t} + av_{x} = \nu v_{xx} \]
(b) Implicit scheme (BTCS)
\[ u_{k}^{n+1} + \frac{a\Delta t}{2\Delta x} \delta_{0} u_{k}^{n+1} - \frac{\nu\Delta t}{\Delta x^2} \delta_{2} u_{k}^{n+1} = u_{k}^{n} \]
\[ v_{t} + av_{x} = \nu v_{xx} \]
(c) Crank-Nicolson Scheme
\[ u_{k}^{n+1} - \frac{\nu\Delta t}{2\Delta x^2} \delta_{2} u_{k}^{n+1} = u_{k}^{n} + \frac{\nu\Delta t}{2\Delta x^2} \delta_{2} u_{k}^{n} \]
\[ v_{t} = \nu v_{xx} \]
(d) DuFort-Frankel Scheme
\[ u_{k}^{n+1} = \frac{2r}{1 + 2r} (u_{k+1}^{n} + u_{k-1}^{n}) + \frac{1 - 2r}{1 + 2r} u_{k}^{n-1} \]
where \( r = \frac{\Delta t}{\Delta x^2} \)
\[ v_{t} = v_{xx} \]
(e) FTFS for a hyperbolic equation
\[ u_{k}^{n+1} = u_{k}^{n} - \frac{a\Delta t}{\Delta x} (u_{k+1}^{n} - u_{k}^{n}) \]
\[ v_{t} + av_{x} = 0 \]
(2). Show that the following scheme is a $O(\Delta t) + O(\Delta x^4)$ approximation of $v_t = \nu v_{xx}$.

$$u_k^{n+1} = u_k^n + r\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)$$

where $r = \nu \frac{\Delta t}{\Delta x^2}$.

(3). Analyze the stability and convergence of the following schemes

(a). FTBS for a hyperbolic equation $v_t + av_x = 0$

$$u_k^{n+1} = R u_{k-1}^n + (1 - R) u_k^n$$

where $R = a \frac{\Delta t}{\Delta x}$.

(b). Higher order scheme for heat equation $v_t = \nu v_{xx}$

$$u_k^{n+1} = u_k^n + r\left(-\frac{1}{12}u_{k-2}^n + \frac{4}{3}u_{k-1}^n - \frac{5}{2}u_k^n + \frac{4}{3}u_{k+1}^n - \frac{1}{12}u_{k+2}^n\right)$$

where $r = \nu \frac{\Delta t}{\Delta x^2}$.

(4). Solve the following problem

$$v_t = \nu v_{xx}, \ x \in (0, 1), \ t > 0, \ \nu = 1/6$$

$$v(x, 0) = \sin 4\pi x$$

$$v(0, t) = 0, v(1, t) = 0, \ t \geq 0$$

using the Crank-Nicolson Scheme. Please use the algorithm introduced in the lecture.

Choose $\Delta x = 0.1, \ \Delta t = 0.01$.

Find the solutions at $t = 0.06, 0.1, 0.9, 10.0$. Compare and contrast your solution with the exact solution and with the solution obtained using the explicit scheme in the last homework (using graphics and write a report on your findings).