03/29/12

\[ \vec{V}_t + A \vec{V}_x = 0. \]

A diagonalizable.

S. \[ S A S^{-1} = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix} \]

Upwind scheme

Define \( \alpha_i^+ = \max(A_i, 0) \)
\( \alpha_i^- = \min(A_i, 0) \)

\[ D^+ = \begin{bmatrix} \lambda_1^+ & 0 & \cdots & 0 \\ 0 & \lambda_2^+ & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^+ \end{bmatrix} \]

\[ D^- = \begin{bmatrix} \lambda_1^- & 0 & \cdots & 0 \\ 0 & \lambda_2^- & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^- \end{bmatrix} \]

\[ A^+ = S D^+ S, \quad A^- = S D^- S \]

\[ \vec{u}_{k+1}^n = \vec{u}_k^n - R A^+ \Delta t \vec{u}_k^n - R A^- \Delta t \vec{u}_k^n \quad R = \frac{\partial f}{\partial x} \]

Lax-Wendroff

\[ \vec{u}_{k+1}^n = \vec{u}_k^n - \frac{k}{2} A \Delta t \vec{u}_k^n - \frac{k^2}{2} A \Delta t^2 \vec{u}_k^n \]

Crank-Nicolson implicit

\[ \vec{u}_k^n + \frac{k}{4} A \Delta t \vec{u}_k^{n+1} = \vec{u}_k^n - \frac{k}{4} A \Delta t \vec{u}_k^n \]
Conservation laws

- Mass
- Momentum
- Energy

Scalar conservation laws:

\[
\frac{2V}{a_t} + \frac{2}{a} \frac{\partial}{\partial x} (F(U)) = 0.
\]

\[F(U)\text{ flux function}\]

\[V_t + aW_x = 0.\]

\[F(U) = aV, \quad V_t + (F(U))_x = 0.\]

\[F(U)\text{ is convex, } F''(U) > 0.\]

\[\text{e.g. } F(U) = \frac{1}{2}V^2.\]

\[V_t + (\frac{1}{2}V^2)_x = 0. \quad \text{inviscid Burger's equation}\]

Nonconservation law form (linearized form):

\[V_t + A(U)V_x = 0, \quad A(U) = F'(U)\]

\[V_t + F'(U)V_x = 0. \quad \text{derivative of } F(U) \text{ with respect to } U.\]

\[V_t + UV_x = 0. \quad \text{inviscid Burger's equation}\]
Define characteristic curves to be the solution of the differential equation

\[ \frac{d(x(t), t)}{dt} = F'(v) = F'(V(x(t), t), t) \]

along the characteristic curve

\[ \frac{dV(x(t), t)}{dt} = V_x(x(t), t) \frac{dx(t)}{dt} + V_t(x(t), t) \]

\[ = V_x(x(x(t), t), t) F'(V(x(t), t), t) + V_t(x(x(t), t), t) . \]

\( V(x(t)) \) solution of \( V_t + F'(v) V_x = 0 \).

\[ \frac{dV(x(t), t)}{dt} = 0 . \]

the solution to PDE is constant.

Note: in the linear case, \( V_t + a V_x = 0 \).

\[ \frac{dx(t)}{dt} = a \text{ characteristic curve} \]

\( x(t) = a t + C \).

the solution of PDE is \( V(x,t) = V_0(x-ax) \).

I.e. \( V(x,0) = V_0(x) \).
If \( V \) is a sufficiently smooth solution to the initial value problem
\[
V_t + F(V)_x = 0.
\]
along with the initial condition \( V(x,0) = V_0(x) \), then \( V \) will satisfy
\[
V(x,t) = V_0(x - \frac{F(V(x,t))}{a} t).
\]

Note: Characteristic curves must satisfy
\[
\frac{dx(t)}{dt} = F'(V(x,t), t).
\]

must be straight line.
\[
x(t) = \frac{F'(V_0)}{a} t + \text{constant}.
\]

\( V_t + aV_x = 0. \) \( x = ct + C. \)

Slope of characteristic curves is \( \frac{1}{a} \).

\( V_t + F(V)_x = 0. \) Slope \( \frac{1}{F'(V_0)} \)
in general, slope is different for various value of \( V_0 \).
For example. \textit{initial-value problem.}

\[ v_t + \left( \frac{1}{2} v^2 \right)_x = 0, \quad x \in [0, 1] \]

\[ I.e. \quad v(x, 0) = v_0(x) = \sin 2\pi x. \]

\[ F(v) = \frac{1}{2} v^2, \quad F'(v) = v \]

characteristic curves

\[ \frac{dx(x(t), t)}{dt} = v(x(t), t). \]

curve starting from \( t=0 \) axis at \( x_0 \) point is given by.

\[ x(t) = F'(v_0) t + \text{constant} \]

\[ = v_0(x_0) t + \text{constant} \]

\[ = \sin 2\pi x_0 t + x_0. \]
When two characteristic curves
\[ x = f'(v_0) t + x_0 \text{ and } x = f'(v_1) t + x, \]
intersect at \((x,t)\), the solution set \((x,t)\)

wants to be both \(v_0\) and \(v_1\),

double valued or more

solution contains discontinuity. \(v_0\) \(v_1\)

**Definition.**

Discontinuous solutions to the scalar
conservation law \(u_t + F(u)_x = 0\) is
considered weak solutions to the PDE.

Consider \(u_t + F(u)_x = 0\), \(x \in \mathbb{R}, t > 0\)

I.C. \(u(x,0) = u_0(x)\).

Define the set of test functions. \(C_0\)
to be the set

\[ \{ \phi \in C^1 : \{ (x,t) \in \mathbb{R} \times [0,\infty) : \phi(x,t) \neq 0 \} \}

\subset \begin{bmatrix} [a,b] \times [0,T] \end{bmatrix} \text{ for some } a, b, T \}

\[ \phi(t) = 0 \]
\[ \phi(a,+) = \phi(b,+) = 0 \]
\[ \phi(x,T) = 0 \]
\[ \phi(x,0) \]
\( \phi \) is once continuously differentiable and zero outside of some rectangle in \( x-t \) space.

A function \( \phi \) that satisfies
\[
\{ (x,t) \in \mathbb{R} \times [0, \infty) : \phi(x,t) \neq 0 \} \subset [a,b] \times [0,T]
\]
said to have compact support in \( \mathbb{R} \times [0, \infty) \).

The support of \( \phi \) is the set on which \( \phi \neq 0 \).

\[
\text{supp}(\phi).
\]

\[\mathcal{U}_t + F(V)x = 0.\]

Multiply PDE by \( \phi \in C_0^\infty \), and integrate with respect to \( x \) from \( -\infty \) to \( +\infty \), with respect to \( t \) for \( 0 \) to \( +\infty \).

\[
0 = \int_0^T \int_{-\infty}^{\infty} \left( \mathcal{U}_t + F(V)x \right) \phi(x,t) \, dx \, dt.
\]

\[
= \int_0^T \int_a^b \left( \mathcal{U}_t + F(V)x \right) \phi(x,t) \, dx \, dt.
\]

\[
= \int_a^b \int_0^T \mathcal{U} \phi(x,t) \, dt \, dx
\]

\[
+ \int_0^T \int_a^b F(V)x \phi(x,t) \, dx \, dt.
\]

\[
= \int_a^b \left[ F(V) \right]_{t=0}^{t=T} - \int_0^T \mathcal{U} \phi \, dt + \int_a^b F(V) \phi \, dx \, dt.
\]

\[
+ \int_0^T \int_a^b \left( F(V) \phi(x,b) - F(V) \phi(x,a) \right) \, dx \, dt.
\]
\[ \int_a^b \left( \nabla \phi(x,t) \right) \cdot \nabla \phi(x,T) \, dx - \int_a^b \phi_t \, dx + \int_0^T \left( (F(v) \phi_x(t) - F(v) \phi_x(t)) - \int_a^b F(v) \phi_x \, dx \right) \, dt \\
\phi(x,T) = \phi(a,t) = \phi(b,t) = 0. \\
\phi(x,0) \neq 0. \\
\int_0^b v(x,0) \phi(x,0) \, dx = \int_a^b \int_0^T u \phi_x \, dx \, dt \\
- \int_0^T \int_a^b F(v) \phi_x \, dx \, dt. \\
\int_a^b \int_0^T \left[ v \phi_t + F(v) \phi_x \right] \, dx \, dt + \int_0^T V(x) \phi_0 \, dx = 0 \\
\phi_0 = \phi(x,0) \quad V_0 = V(x,0). \\
\]

If \( V \) is a classical solution to the initial value problem \( V_t + F(v) \phi_x = 0, \) \( V(x,0) = V_0(x) \), then \( V \) will satisfy equation (*) for all \( \phi \in C^1 \).

If \( V \) is continuously differentiable with respect to \( x \) and \( t \) and satisfies equation (*) for all \( \phi \in C^1 \), then \( V \) is a classical solution to the initial value problem.
Definition:
If \( V \) satisfies equation (\( \star \)) for all \( \phi \in C_0 \)
\( \phi \), \( V \) is said to be a weak solution to the initial value problem
\( V_t + F(V)_x = 0, \ V(x,0) = V_0(x) \).