Numerical dissipation and dispersion.

\[ V(x,t) = V e^{i(\omega t + \beta x)} \]

- \( \omega \): frequency of the wave
- \( \beta \): wave number

\[ V_t + \alpha V_x = 0 \quad \text{wave equation} \]

- \( \omega = -i \beta^2 \): dispersion relation
- wave doesn't move.
- decays with time, dissipation.

\[ V(x,t) = V e^{i(\omega t + \beta x)} \]

- no decay of the amplitude.
- wave move with a speed independent of frequency and wave number.

Numerically introduced decay and propagation is numerically dissipation and dispersion.
dissipation: Fourier modes decay with time

 dispersion: modes of different wave number propagate at different speed.

 PDE contains only even ordered x derivative
  - dissipative
    \[ V_t = D V_{xx} \]

 PDE contains only odd ordered x derivative
  - non dissipative
    - dispersive if order is greater than 1.
    \[ V_t + a V_x = 0 \quad \text{and} \quad V_t = D V_{xxxx} \]

 Hyperbolic equation.

 FD. \[ u_t + a u_x = D u_{xx} + \cdots \]
    numerical dissipation.

 FD. \[ u_t + a u_x = A u_{xxxx} + \cdots \]
    numerical dispersion.
Upwind: \( D_n = \frac{1}{2} a \Delta x \left( 1 - R_1 \right) \).

L-F: \( D_n = \frac{1}{2} \frac{\Delta x^2}{\Delta t} \left( 1 - R^2 \right) \).

L-F is more dissipative than upwind.

Centered implicit:
\( D_n = \frac{1}{2} a \Delta x \).

First-order schemes introduce numerical dissipation.

Leapfrog: \( \lambda = \frac{a}{6} (R^2 - 1) \Delta x^2 \).

L-W: \( \lambda = \frac{a}{6} (R^2 - 1) \Delta x^2 \).

C-N: \( \lambda = -\frac{1}{6} a \Delta x^2 \).

C-N is more dispersive than L-W.
Tridiagonal system of equations

\[ A\mathbf{u} = \mathbf{d} \]

\[ A = \begin{pmatrix}
    b_1 & c_1 & 0 & \cdots \\
    a_2 & b_2 & c_2 & 0 & \cdots \\
    0 & a_3 & b_3 & c_3 & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}_{N \times N} \]

\[ \mathbf{u} = \begin{pmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_{N-1} \\
    u_N
\end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix}
    d_1 \\
    d_2 \\
    \vdots \\
    d_{N-1} \\
    d_N
\end{pmatrix} \]

\[ U_{i+1} = A_i u_i + B_i, \quad i = 1, \ldots, N-1 \]

\[ A_{N-1} = -\frac{a_N}{b_N}, \quad B_{N-1} = \frac{d_N}{b_N} \]

\[ A_{i-1} = \frac{-a_i}{A_i c_i + b_i}, \quad B_{i-1} = \frac{d_i - B_i c_i}{A_i c_i + b_i}, \quad i = 2, \ldots, N-1 \]

1. \[ \{ a_1, b_1 \} \quad \ldots \quad \{ a_{N-1}, b_{N-1} \} \]
2. \[ u_i = \frac{d_i - B_i c_i}{A_i c_i + b_i} \]
3. \[ u_{i+1} = A_i u_i + B_i. \quad \Rightarrow \quad \mathbf{u} \]