Homework # 4 Solutions

Section 3.2 : 7(b), 14, 18, 20 (a and b) – whichever one you did, 28

Section 3.3: 11(a), 13, 15, 22

Section 3.2

7. (b) A matrix $X$ satisfies the equation $B + (A + X)^T = C$ if and only if

$$(A + X)^T = C - B$$

$$A + X = ((A + X)^T)^T = (C - B)^T$$

$$X = (C - B)^T - A = C^T - B^T - A$$

Thus

$$X = \begin{bmatrix} 0 & 1 & 3 \\ -2 & 7 & 5 \\ 3 & 4 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 4 \\ -3 & 1 & -7 \\ -5 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 5 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -10 & 2 & -4 \\ 1 & 2 & 7 \\ 10 & 1 & -1 \end{bmatrix}.$$

14. $X = (C - B)^{-1}AB = \frac{1}{22} \begin{bmatrix} -5 & -7 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 10 & -5 \\ 18 & -7 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -88 & 37 \\ 66 & -29 \end{bmatrix}$

18. $(AC^{-1})^{-1}(AC^{-1})AC^{-1}AD^{-1} = CA^{-1}AC^{-1}AD^{-1} = CD^{-1}$

$$= \begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 6 & 4 \\ -3 & -1 \end{bmatrix}$$

20. (a) Given that $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$ it follows that $5A^T = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}^{-1} = (-1)\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$

and so $A = \frac{1}{5} \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -2 & 5 \\ 5 & -3 \end{bmatrix}$.

(b) Given $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$, we have $I + 2A = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{1}{13} \begin{bmatrix} -5 & 2 \\ 4 & 4 \end{bmatrix}$ and so it follows that

$$A = \frac{1}{2} \left( \frac{1}{13} \begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \frac{1}{13} \begin{bmatrix} -9 & -1 \\ 2 & -6 \end{bmatrix}.$$

28.

(a) The matrix $uv^T = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$ has the property that

$$u^Tv = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = 17 = tr \left( \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -6 \end{bmatrix} \right) = tr(uv^T)$$.  

(b) \( \mathbf{u} \cdot \mathbf{Av} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 & -1 \\ 1 & 5 & 0 \\ -2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -4 \\ 12 \end{bmatrix} = -6 + 8 + 48 = 50 \)

\( \mathbf{A}^T \mathbf{u} \cdot \mathbf{v} = \left( \begin{bmatrix} 3 & 1 & -2 \\ 2 & 5 & 4 \\ -1 & 0 & 6 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 12 \\ 21 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = -1 - 12 + 63 = 50 \)

Section 3.3

11. (a) Start with the partitioned matrix \( [\mathbf{A} \mid \mathbf{I}] \).

\[
\begin{bmatrix}
3 & 4 & -1 & 1 & 0 \\
1 & 0 & 3 & 0 & 1 \\
2 & 5 & -4 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Interchange rows 1 and 2.

\[
\begin{bmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
3 & 4 & -1 & 1 & 0 & 0 \\
2 & 5 & -4 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Add \(-3\) times row 1 to row 2. Add \(-2\) times row 1 to row 3.

\[
\begin{bmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
0 & 4 & -10 & 1 & -3 & 0 \\
0 & 5 & -10 & 0 & -2 & 1 \\
\end{bmatrix}
\]

Add \(-1\) times row 2 to row 3.

\[
\begin{bmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
0 & 4 & -10 & 1 & -3 & 0 \\
0 & 1 & 0 & -1 & 1 & 1 \\
\end{bmatrix}
\]

Add \(-4\) times row 3 to row 2, then interchange rows 2 and 3.

\[
\begin{bmatrix}
1 & 0 & 3 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 & 1 \\
0 & 0 & -10 & 5 & -7 & -4 \\
\end{bmatrix}
\]

Multiply row 3 by \(-\frac{1}{10}\), then add \(-3\) times the new row 3 to row 1.

\[
\begin{bmatrix}
1 & 0 & 0 & \frac{3}{2} & -\frac{11}{10} & -\frac{6}{5} \\
0 & 1 & 0 & -1 & 1 & 1 \\
0 & 1 & 1 & \frac{3}{2} & \frac{7}{10} & \frac{2}{5} \\
\end{bmatrix}
\]
As in the inversion algorithm, we start with the partitioned matrix \([A \mid I]\) and perform row operations aimed at reducing the left side to its reduced row echelon form \(R\). If \(A\) is invertible, then \(R\) will be the identity matrix and the matrix produced on the right side will be \(A^{-1}\). In the more general situation the reduced matrix will have the form \([R \mid B] = [BA \mid BI]\) where the matrix \(B\) on the right has the property that \(BA = R\). [Note that \(B\) is the product of elementary matrices and thus is always an invertible matrix, whether \(A\) is invertible or not.]

\[
\begin{bmatrix}
1 & 2 & 3 & | & 1 & 0 & 0 \\
0 & 0 & 1 & | & 0 & 1 & 0 \\
1 & 2 & 4 & | & 0 & 0 & 1
\end{bmatrix}
\]

Add \(-1\) times row 1 to row 3.

\[
\begin{bmatrix}
1 & 2 & 3 & | & 1 & 0 & 0 \\
0 & 0 & 1 & | & 0 & 1 & 0 \\
0 & 0 & 1 & | & -1 & 0 & 1
\end{bmatrix}
\]

Add \(-1\) times row 2 to row 3.

\[
\begin{bmatrix}
1 & 2 & 3 & | & 1 & 0 & 0 \\
0 & 0 & 1 & | & 0 & 1 & 0 \\
0 & 0 & 0 & | & -1 & -1 & 1
\end{bmatrix}
\]

Add \(-3\) times row 2 to row 1.

\[
\begin{bmatrix}
1 & 2 & 0 & | & 1 & -3 & 0 \\
0 & 0 & 1 & | & 0 & 1 & 0 \\
0 & 0 & 0 & | & -1 & -1 & 1
\end{bmatrix}
\]

The reduced row echelon form of \(A\) is \(R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\), and the matrix \(B = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}\) has the property that \(BA = R\).

15. (a) \(A^{-2} = (A^{-1})^2 = \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}^2 = \begin{bmatrix} 9 & -5 \\ -25 & 14 \end{bmatrix}\)

(b) \(p(A) = A + 2I = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 5 & 4 \end{bmatrix}\)

(c) \(p(A) = A^2 - 2A + I = \begin{bmatrix} 14 & 5 \\ 25 & 9 \end{bmatrix} - 2 \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 15 & 6 \end{bmatrix}\)

22. The matrix \(A = \begin{bmatrix} -c & -1 \\ 1 & c \end{bmatrix}\) is invertible if and only if \(\det(A) = -c^2 + 1 \neq 0\), i.e. if and only if \(c \neq \pm 1\).