10. Population Mean

\[ \bar{X} \] \text{ sample \ mean} \n
\[ \mu_{\bar{X}} = \mu, \quad \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \]

If population has normal distribution:
\[ X \sim N(\mu, \frac{\sigma}{\sqrt{n}}) \] \text{ any sample size}

If population not normal, but \( n \geq 30 \)
\[ X \sim \text{approx.} \quad N(\mu, \frac{\sigma}{\sqrt{n}}) \]
\text{(central limit theorem)}

10.2. Hypothesis testing about \( \mu \).

1. Assume \( \sigma \) is known. (not practical)

\[ H_0: \mu = \mu_0 \quad \text{vs.} \quad \left\{ \begin{array}{l}
H_1: \mu > \mu_0 \\
H_1: \mu < \mu_0 \\
H_1: \mu \neq \mu_0
\end{array} \right. \]

\[ Z \text{-test statistic} \quad Z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \]

obtain the p-value \[ Z \sim N(0,1) \text{ standard normal} \]
For $H_1: M > M_0$ (to the right),
$p$-value = area to the right of $z$

For $H_1: M < M_0$ (to the left),
$p$-value = area to the left of $z$

For $H_1: M \neq M_0$ (two sided),
$p$-value = $2 \times$ area to the left of $-|z|$

With $p$-value and required $\alpha$,
If $p$-value $> \alpha$, $\rightarrow$ accept $H_0$
If $p$-value $\leq \alpha$ $\rightarrow$ reject $H_0$. Take it,
"Statistically significant."

1. the value of $\sigma$ is unknown (practical)
   Sample standard deviation $s$
The distribution of that $T$ variable is called a **Student's $t$-distribution with $n-1$ degree of freedom**.

Obtain the $p$-value as follows:

For $H_0: \mu = \mu_0$ (to the right):

- $p$-value = area to the right of $t$ with df. = $n-1$

For $H_0: \mu < \mu_0$ (to the left):

- $p$-value = area to the left of $t$ with df. = $n-1$

For $H_0: \mu \neq \mu_0$ (two sided):

- $p$-value = 2 x area to the left of $|t|$ with df. = $n-1$
Features of \( t \)-distribution:
1. Symmetric, Bell shaped with mean = 0
2. Flatter with heavier tails than \( N(0,1) \)

\[
\begin{align*}
&\begin{array}{c}
t(x) \\
\end{array} \quad \begin{array}{c}
N(0,1) \\
\end{array} \\
&\quad \quad \quad \quad \quad \quad 0
\end{align*}
\]

3. \( df \uparrow \), \( t \)-distribution becomes more like \( N(0,1) \).
   If \( df > 120 \), we use \( N(0,1) \).

4. Table IV gives the area to the right of some \( t \) values for \( df = 1, 2, \ldots, 120 \).

5. For \( df \geq 121 \) (\( df > 120 \)), use \( z \)-line. The \( z \)-line gives the area to the right of some \( z \) values.
Example: \( n = 10 \) \( \text{df} = n-1 = 9 \)

\[
P(t > 0.261) = 0.4 \\
P(t > 1.833) = 0.1 \\
P(t < 1.833) = 0.9 \\
P(t < -1.833) = P(t > 1.833) = 0.05
\]

\[
P(t > 2) \text{ if } 1.83 < a < 2.262 \\
\Rightarrow 0.025 < P(t > 2) < 0.05
\]

10.9 Confidence interval estimation for \( \mu \).

\( \bar{X} \) is a point estimate of \( \mu \).

(1) \( \sigma \) is known (not practical)

\[
\bar{X} \pm z^* \left( \frac{\sigma}{\sqrt{n}} \right)
\]

where \( z^* \) is the appropriate percentile of the \( N(0, 1) \) distribution.

(1-\( \alpha \)) 100\% \( z^* \) \( P(z < z^*) = 1 - \frac{\alpha}{2} \).
2. If  is unknown (most of the case)

\[ X \pm t^* \left( \frac{S}{\sqrt{n}} \right) \]

Where \( t^* \) is the appropriate percentile of the \( t \)-distribution with \( n-1 \) df.

\[ t(n-1) \]

\[ P(t < t^*) = \left( \frac{\alpha}{2} \right) \]

**Margin of error**

1. If known. \( E = z^* \left( \frac{S}{\sqrt{n}} \right) \)

2. If unknown. \( E = t^* \left( \frac{S}{\sqrt{n}} \right) \).

Let's do it! 10.3 (to the right)

**Hypothesis:**

- \( H_0: \mu = 7.06 \) vs. \( H_1: \mu > 7.06 \)
- \( n = 15 \). \( \overline{X} = 8.43 \). \( S = 4.32 \)
- \( \mu_0 = 7.06 \).

**t-test statistic.**

\[ t = \frac{\overline{X} - \mu_0}{S/\sqrt{n}} = 1.23 \]
(a).

\[ t(14) \]

0 \[ \leq \] \( T \) \[ \leq \] \( t(14) \)

\[ \Rightarrow \]

\[ 0.1 < \text{p-value} < 0.2 \]

(b).

\text{p-value (Table IV)}

\[ 0.868 < t < 1.345 \]

\[ \Rightarrow \]

\[ 0.1 < \text{p-value} < 0.2 \]

(c).

\[ \alpha = 0.1 \]

\[ \Rightarrow \]

\[ \text{p-value} > \alpha \]

\[ \Rightarrow \]

accept H0.

(d).

Our data doesn't show enough evidence for us to conclude that the study time is more than 7.66 hours.

(e).

Construct 90% confidence interval for \( \theta \).

\[ \bar{x} = 8.43 \]

\[ \frac{9}{\sqrt{n}} = \frac{4.32}{\sqrt{15}} = 1.12 \]
Table IV

\( t^* = 1.761 \)  
\( \bar{x} \pm t^* \left( \frac{s}{\sqrt{n}} \right) \)

\( 8.93 \pm 1.761 \times 0.12 \)

\( \Rightarrow (6.46, 10.4) \)  
90%