1. (10 points) Find the general solution of the differential equation.

\[ xy + 4y^2 - x^2 y' = 0 \]

Solution:

\[ y' = \frac{4y^2 + xy}{x^2} = 4\left(\frac{y}{x}\right)^2 + \frac{y}{x} \]

Homogeneous DE: Use the transformation \( v = \frac{y}{x} \).

\[ y = xv \Rightarrow y' = v + xv' \]

Substitute into the DE

\[ v + xv' = 4v^2 + v \Rightarrow xv' = 4v^2 \Rightarrow \frac{dv}{v^2} = \frac{4dx}{x} \]

Integrate both sides

\[ -\frac{1}{v} = 4\ln x - c \Rightarrow v(x) = \frac{1}{c - 4\ln x} \]

\[ y(x) = \frac{x}{c - 4\ln x} \]

2. (15 points) The differential equation is

\[ (\cos x - \ln y)dx + \left(e^y - \frac{x}{y}\right)dy = 0 \]

(a) (5 points) Verify that the given differential equation is exact.

(b) (10 points) Solve the given differential equation.

Solution:

(a)

\[ M(x, y) = \cos x - \ln y, \quad N(x, y) = e^y - \frac{x}{y} \]

The differential equation is exact

\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow -1 = -1 \frac{1}{y} \]

(b) The solution, \( F(x, y) = c \), can be obtained by

\[ \partial F = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M(x, y)dx + N(x, y)dy = 0 \]
\[
\frac{\partial F}{\partial x} = \cos x - \ln y \Rightarrow F(x, y) = \int (\cos x - \ln y) \, dx = \sin x - x \ln y + g(y)
\]
\[
\frac{\partial F}{\partial y} = -\frac{x}{y} + g'(y) = e^y - \frac{x}{y} \Rightarrow g'(y) = e^y \Rightarrow g(y) = \int e^y \, dy = e^y + c_1
\]
\[
F(x, y) = \sin x - x \ln y + e^y + c_1 = c \Rightarrow \sin x - x \ln y + e^y = C
\]

3. (15 points) The differential equation is
\[
\frac{dy}{dx} + 6x y - 4y^2 = \frac{1}{x^2}
\]
(a) (5 points) Show that the substitution \( y(x) = x^{-1} + u(x) \) transform the differential equation into the Bernoulli equation.
(b) (10 points) Solve the resulting Bernoulli equation.

Solution:
(a) \( y(x) = x^{-1} + u(x) \Rightarrow y' = -x^{-2} + u' \)

DE becomes:
\[-x^{-2} + u' + 6x^{-1}(x^{-1} + u) - 4(x^{-1} + u)^2 = x^{-2}\]
\[-x^{-2} + u' + 6(x^{-2} + x^{-1}u) - 4(x^{-2} + u^2 + 2x^{-1}u) = x^{-2}\]
\[u' - 2x^{-1}u - 4u^2 = 0 \Rightarrow u' - \frac{2}{x}u = 4u^2\]

(b) For the solution of Bernoulli equation use the substitution \( v = u^{-1} \).
\[v' = -u^{-2}u' \Rightarrow u' = -\frac{v'}{v^2}\]
\[-v'v^{-2} - \frac{2}{x}v^{-1} = 4v^{-2} \Rightarrow v' + 2x^{-1}v = -4\]

Bernoulli equation is transformed into Linear First-Order differential equation.

Step 1: Find the integrating factor
\[\rho(x) = e^{\int \frac{2}{x} \, dx} = e^{2\ln x} = x^2\]

Step 2: Multiply the both sides of DE by integrating factor
\[x^2v' + 2xv = -4x^2\]

Step 3:
\[D_x(x^2v) = -4x^2\]

Step 4: Integrate the both sides
\[x^2v = -\frac{4x^3}{3} + c \Rightarrow v = -\frac{4x}{3} + cx^{-2}\]
\[v = u^{-1} \Rightarrow u = v^{-1} = \frac{1}{-\frac{4x}{3} + cx^{-2}} y = \frac{1}{x} + \frac{1}{-\frac{4x}{3} + cx^{-2}}\]
4. (10 points)

The time rate of change of a rabbit population $P$ is proportional to the square root of $P$. At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 40 rabbits per month. How many rabbits will there be one year later?

Solution: The differential equation

$$P' = k\sqrt{P}$$

$$P(0) = 100 \quad P'(0) = 40$$

Using the DE and initial conditions, the value of $k$ can be obtained.

$$P'(0) = k\sqrt{P(0)} \Rightarrow 40 = k\sqrt{100} = k10 \Rightarrow k = 4$$

$$P' = 4\sqrt{P} \Rightarrow \frac{dP}{\sqrt{P}} = 4 dt \Rightarrow \int P^{-1/2}dP = \int 4dt \Rightarrow 2P^{1/2} = 4t + c$$

$$2P^{1/2}(0) = 2\sqrt{100} = 20 = c$$

$$P(t) = (2t + 10)^2 \Rightarrow P(12) = (2.12 + 10)^2 = 1156$$