1. Chapter 1.6 Problem 11, Page 71

Find general solution of the differential equation.

\[(x^2 - y^2) \ y' = 2xy\]

**Solution:**

\[y' = \frac{2xy}{x^2 - y^2} = \frac{2xy}{x^2 - y^2} \left(\frac{x}{x}\right) = \frac{2x}{1 - \left(\frac{y}{x}\right)^2}\]

Homogenous DE: Use the substitution \(u = y/x\) to solve the Homogenous DE.

\[y = xu \implies \frac{dy}{dx} = u + x\frac{du}{dx}\]

After substituting the transformation into the DE, we get

\[u + x\frac{du}{dx} = \frac{2u}{1 - u^2} \implies x\frac{du}{dx} = \frac{2u - u(1 - u^2)}{1 - u^2} = \frac{u^3 + u}{1 - u^2}\]

It is transformed into the Separable DE.

\[\int \frac{1 - u^2}{u^3 + u} \, du = \int \frac{dx}{x}\]

Using the partial fraction:

\[\frac{1 - u^2}{u^3 + u} = \frac{1 - u^2}{u(u^2 + 1)} = A + \frac{Bu}{u^2 + 1}\]

To find the values of \(A\) and \(B\), set the coefficients of the polynomial terms individually.

\[1 - u^2 = A(u^2 + 1) + (Bu)u \implies A = 1 \quad & \quad A + B = -1\]

\[\int \left(\frac{1}{u} - \frac{2u}{u^2 + 1}\right) \, du = \int \frac{dx}{x}\]

Integrating both sides of the equation gives:

\[\ln u - \ln(u^2 + 1) = \ln x + \ln C \implies u = Cx(u^2 + 1) \implies y = C(y^2 + x^2)\]

2. Chapter 1.6 Problem 18, Page 71

Find general solution of the differential equation.

\[(x + y)y' = 1\]

**Solution:** By using the substitution \(u = x + y\), \(\Rightarrow u' = 1 + y'\)

\[y' = \frac{1}{x + y} \implies u' - 1 = \frac{1}{u}\]

\[\frac{du}{dx} = \frac{1 + u}{u} \implies \int \frac{u}{1 + u} \, du = \int \left(1 - \frac{1}{1 + u}\right) \, du = \int dx\]

Integration of both sides of equations gives us

\[u - \ln(1 + u) = x + c \implies x + y - \ln(1 + x + y) = x + c\]

\[y = \ln(1 + x + y) + c\]
3. Chapter 1.6 Problem 24, Page 71
Find general solution of the differential equation.

\[ 2xy' + y^3e^{-2x} = 2xy \]

Solution:

\[ y' - y = -\frac{e^{-2x}}{2x}y^3 \]

Bernoulli DE is solved by using the substitution \( u = y^{1-3} = y^{-2} \).

\[ u' = -2y^{-3}y' \]

The substitution transform the equation into the Linear DE.

\[ u' + 2u = \frac{e^{-2x}}{x} \]

Step1: Find the integrating factor \( \rho(x) \) where \( P(x) = 2 \) and \( Q(x) = e^{-2x}/x \).

\[ \rho(x) = e^{\int 2dx} = e^{2x} \]

Step2: Multiply the both sides of the DE by \( \rho(x) \).

\[ e^{2x}u' + 2e^{2x}u = \frac{1}{x} \]

Step3:

\[ D_x(e^{2x}u) = \frac{1}{x} \]

Step4: Integrate the both sides:

\[ e^{2x}u = \int \frac{1}{x} dx = \ln x + c \Rightarrow u = e^{-2x}(\ln x + c) \Rightarrow y = \frac{e^{2x}}{(\ln x + c)} \]

4. Chapter 1.6 Problem 51, Page 72
Find a general solution of second-order differential equation. Assume \( x, y, \) and/or \( y' \)
positive where helpful (as in Example 11).

\[ y''' = 2y(y')^3 \]

Hint: Use the substitution \( p = y' \) defined in equation (36).

Solution:

\[ y' = p, \quad y'' = pp' = p(dp/dy) \]

Substitute into the DE gives:

\[ p\frac{dp}{dy} = 2yp^3 \quad \Rightarrow \quad \frac{dp}{p^2} = 2ydy \]

Seperable equation is solved by integrating the both sides of equation.

\[ \int \frac{dp}{p^2} = \int 2ydy \Rightarrow -\frac{1}{p} = y^2 + c \Rightarrow p = \frac{dy}{dx} = -\frac{1}{y^2 + c} \]

\( (y^2 + c)dy = -dx \Rightarrow \frac{y^3}{3} + cy = -x + d \Rightarrow y^3 + 3x + Ay + B = 0 \)
5. Chapter 1.6 Problem 58, Page 72

Solve the differential equation

\[ x \frac{dy}{dx} - 4x^2 y + 2y \ln y = 0 \]

*Hint:* The substitution \( v = \ln y \) transform the differential equation into the Linear First Order Differential Equation.

**Solution:**

\[ v = \ln y \Rightarrow y = e^v \Rightarrow \frac{dy}{dx} = e^v \frac{dv}{dx} \]

Substitute these into the DE gives:

\[ xe^v \frac{dv}{dx} - 4x^2 e^v + 2e^v v = 0 \]

\[ \frac{dv}{dx} + \frac{2}{x} v = 4x \]

The resulting Linear 1st order differential equation is solved by using the integrating factor \( \rho(x) \). Step1: Find the integrating factor.

\[ \rho(x) = e^{ \int \frac{2}{x} dx} = e^{2 \ln x} = e^{\ln x^2} = x^2 \]

Step2: Multiply the both sides of DE by integrating factor \( \rho(x) \).

\[ x^2 \frac{dv}{dx} + x^2 \frac{2}{x} v = (4x)x^2 \]

Step3: Left-hand side of DE is equal to \( D_x(x^2 v) \).

\[ D_x(x^2 v) = 4x^3 \]

Step4: Integrating the both sides of equation gives

\[ x^2 v = \int 4x^3 dx = x^4 + c \Rightarrow v = x^2 + cx^{-2} \]

\[ y = e^{x^2 + cx^{-2}} \]

6. Chapter 1.6 Problem 59, Page 72

Solve the differential equation

\[ \frac{dy}{dx} = \frac{x - y - 1}{x + y + 3} \]

by finding \( h \) and \( k \) so that the substitutions \( x = u + h \), \( y = v + k \) transform it into the homogeneous equation

\[ \frac{dv}{du} = \frac{u - v}{u + v} \]

**Solution:**

\[ x = u + h \Rightarrow dx = du, \quad y = v + k \Rightarrow dy = dv \]

\[ \frac{dv}{du} = \frac{u - v + h - k - 1}{u + v + h + k + 3} = \frac{u - v}{u + v} \]
$h$ and $k$ should satisfy the equations $h - k = 1$ and $h + k = -3$ to transform the given DE into the homogeneous equation. So $h = -1$ and $k = -2$. The substitutions should be $x = u - 1$ and $y = v - 2$.

$$\frac{dv}{du} = \frac{u - v}{u + v} = \frac{\frac{u - v}{u}}{\frac{u + v}{u}}$$

$$\frac{dv}{du} = \frac{1 - v}{1 + v}$$

To solve Homogenous DE, use the substitution $w = v/u$.

$$v = uw \Rightarrow \frac{dv}{du} = w + u \frac{dw}{du}$$

$$w + u \frac{dw}{du} = \frac{1 - w}{1 + w} \Rightarrow u \frac{dw}{du} = \frac{1 - w - w(1 + w)}{1 + w} = \frac{-w^2 + 2w - 1}{1 + w}$$

$$\int \frac{1 + w}{w^2 + 2w - 1} dw = \int \frac{-1}{u} du$$

$$\frac{1}{2} \ln(w^2 + 2w - 1) = -\ln u + \ln C \Rightarrow \ln(w^2 + 2w - 1) = -2\ln u + 2\ln C = -\ln u^2 + \ln C^2$$

$$(w^2 + 2w - 1)u^2 = c \Rightarrow v^2 + 2uv - u^2 = c$$

$$(y + 2)^2 + 2(x + 1)(y + 2) - (x + 1)^2 = c \Rightarrow y^2 + 2xy - x^2 + 2x + 6y = c$$

7. Chapter 1.6 Problem 63, Page 72

The equation $dy/dx = A(x)y^2 + B(x)y + C(x)$ is called a **Riccati equation**. Suppose that one particular solution $y_1(x)$ of this equation is known. Show that the substitution

$$y = y_1 + \frac{1}{v}$$

transforms the Riccati equation into the linear equation

$$\frac{dv}{dx} + (B + 2Ay_1)v = -A$$

**Solution:** First remind ourselves that the particular solution of the DE satisfies the DE.

That means

$$\frac{dy_1}{dx} = A(x)y_1^2 + B(x)y_1 + C(x)$$

From the substitution

$$y = y_1 + v^{-1} \Rightarrow y' = y'_1 - v^{-2}v'$$

After substituting

$$\frac{dy_1}{dx} - v^{-2}\frac{dv}{dx} = A(x)(y_1 + v^{-1})^2 + B(x)(y_1 + v^{-1}) + C(x)$$

$$= A(x)(y_1^2 + v^{-2} + 2y_1v^{-1}) + B(x)(y_1 + v^{-1}) + C(x)$$

If we use the DE for the particular solution, we will get

$$-v^{-2}\frac{dv}{dx} = A(x)(v^{-2} + 2y_1v^{-1}) + B(x)v^{-1}$$

$$\frac{dv}{dx} = A(x)(-1 - 2y_1v) - B(x)v = -(B + 2Ay_1)v - A$$

So;

$$\frac{dv}{dx} + (B + 2Ay_1)v = -A$$