Homework Assignment 3, Sample Test 2

June 30, 2011

Note: The exam is closed-book.

1. (15 points) Are the following statements true or false? Justify your answer briefly. (For each problem, one point for answering true or false correctly and two points for the justification.)

(i) If a function \( f(x, y) \) is differentiable at a point \((a, b)\), then its partial derivatives \( f_x \) and \( f_y \) exist and are continuous on a small disk centered at the point \((a, b)\).

Answer: False. If a function \( f \) is differentiable, then its partial derivatives \( f_x \) and \( f_y \) exist but they may be not continuous. (A counterexample is \( f(x, y) = (x^2 + y^2) \sin(x^2 + y^2)^{-1/2} \), which is differentiable everywhere but whose partials are not continuous at the origin.)

(ii) If \( f(x, y) \) is a continuous function on a bounded (but not necessarily closed) region \( R \), then \( f \) has a global maximum at some point in \( R \).

Answer: False. Since \( R \) is not closed, the global maximum may be at a boundary point of \( R \) that is not in \( R \).

(iii) If a smooth function \( f(x, y) \) has a maximum or minimum subject to a constraint \( g(x, y) = c \) at a point \( P_0 \), then \( P_0 \) must satisfy the equation \( \nabla f = \lambda \nabla g \).

Answer: False. \( P_0 \) could be at a point where \( \nabla g = 0 \) or \( \nabla g \) does not exist.

(iv) When computing double integral \( \iint_R f \, dA \) over a rectangle \( R = [a, b] \times [c, d] \), where \( a, b, c, d \) are all constants, then \( \iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy \).

Answer: True. \( a \leq x \leq b \) and \( c \leq y \leq d \) in \( R \) so \( \iint_R f \, dA = \int_c^d \int_a^b f(x, y) \, dx \, dy \).

(v) If \( W_1 \) and \( W_2 \) are solid regions with the volume of \( W_1 \) greater than the volume of \( W_2 \), then \( \iiint_{W_1} f \, dV > \iiint_{W_2} f \, dV \).

Answer: False. For example, if \( f = 0 \), then \( \iiint_{W_1} f \, dV = \iiint_{W_2} f \, dV \) instead of \( \iiint_{W_1} f \, dV < \iiint_{W_2} f \, dV \).

2. (15 points. 3 points each.) Multiple choice questions.

(i) Let \( w = xy \), where \( x = 5 \sin t \) and \( y = -8 \cos t \). Find \( \frac{dw}{dt} \).

(a) \( \frac{dw}{dt} = -40(\cos^2 t - \sin^2 t) \)

(b) \( \frac{dw}{dt} = -40 \)

(c) \( \frac{dw}{dt} = -40(2 - \sin^2 t) \)

(d) \( \frac{dw}{dt} = -40(\cos^2 t + \sin t) \)

(e) \( \frac{dw}{dt} = -40(\cos t + \sin t) \)

Answer: (a)

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}
\]

\[
\frac{\partial w}{\partial x} = y = -8 \cos t, \quad \frac{dx}{dt} = 5 \cos t, \quad \frac{\partial w}{\partial y} = x = 5 \sin t, \quad \frac{dy}{dt} = 8 \sin t
\]

\[
\frac{dw}{dt} = -40 \cos^2 t + 40 \sin^2 t = -40(\cos^2 t - \sin^2 t)
\]

(ii) Find the directional derivative of the function \( f(x, y) \) at point \( P \) in the direction of \( v \), where \( f(x, y) = x^3 - y^3, \ P(3, 2), \ v = \sqrt{2}(i + j) \).
(a) $7.5\sqrt{2}$  
(b) $28.5\sqrt{2}$  
(c) $15.0\sqrt{2}$  
(d) $10.5\sqrt{2}$  
(e) $-28.5\sqrt{2}$

**Answer: (a)**  
$D_2 f(x, y) = \nabla f(3, 2) \cdot \vec{v}/\|\vec{v}\|$  
$\nabla f(x, y) = f_x i + f_y j = 3x^2 i - 3y^2 j$  
$\nabla f(3, 2) = 27i - 12j$  
$\nabla f(3, 2) \cdot \vec{v}/\|\vec{v}\| = \left(27\left(\frac{\sqrt{2}}{2}\right)\right) - \left(12\left(\frac{\sqrt{2}}{2}\right)\right) = 7.5\sqrt{2}$

(iii) Find an equation of the tangent plane to the surface $g(x, y) = x^2 - y^2$ at the point $(7, 2, 45)$.

(a) $14(x - 7) - 4(y - 2) + (z - 45) = 0$  
(b) $-14(x - 7) + 4(y - 2) - (z - 45) = 0$  
(c) $7(x - 7) - 2(y - 2) - (z - 45) = 0$  
(d) $14(x - 7) + 2(y - 2) - (z - 45) = 0$  
(e) $14(x - 7) - 4(y - 2) - (z - 45) = 0$

**Answer: (e)**  
The gradient at the point $(7, 2, 45)$ is normal to the level surface $g(x, y) = 45$.  
$\nabla g(x, y) = 2x i - 2y j$, $\nabla g(7, 2) = 14i - 4j$  
The equation of the tangent plane is  
$g_x(7, 2)(x - 7) + g_y(7, 2)(y - 2) - (z - 45) = 0 \Rightarrow 14(x - 7) - 4(y - 2) - (z - 45) = 0$.

(iv) Use a double integral in polar coordinates to find the volume of the solid inside the hemisphere $z = \sqrt{81 - x^2 - y^2}$ but outside the cylinder $x^2 + y^2 = 49$.

(a) $\frac{3}{2}32^{3/2}$  
(b) $\frac{3}{2}32^{3/2}$  
(c) $\frac{3}{2}32^{3/2}\pi$  
(d) $\frac{3}{2}32^{3/2}\pi$  
(e) $\frac{3}{2}32^{3/2}\pi$

**Answer: (d)**  
$z = \sqrt{81 - x^2 - y^2} = \sqrt{81 - (x^2 + y^2)} = \sqrt{81 - r^2}$,  
and the volume is  
$f_0^{2\pi} f_{\sqrt{81 - r^2}}^{9} r dr d\theta$.  
Let $u = 81 - r^2$, $du = -2r dr$, $dr = -\frac{du}{2r}$. The volume is therefore  
$f_{27}^{2\pi} \left(\frac{2}{3}\right) \left(81 - r^2\right)^{\frac{3}{2}} dr d\theta = f_{0}^{2\pi} \left(\frac{2}{3}\right) \left(32\right)^{\frac{3}{2}} d\theta = \frac{1}{\left(3\right)} \left(32\right)^{\frac{3}{2}} \pi = \frac{2}{\left(3\right)} \left(32\right)^{\frac{3}{2}} \pi$.

(v) Find the average value of $f(x, y) = e^{x+y}$ over the region $R$, where $R$ is a triangle with vertices $(0, 0)$, $(0, 14)$, and $(14, 14)$.

(a) $(e^{14} - 1)^{2}/2$  
(b) $(e^{14} + 1)^{2}/2$  
(c) $(e^{14} + 1)^{2}/196$  
(d) $(e^{14} - 1)^{2}/196$  
A. $(e^{14} - 1)^{2}$

**Answer: (d)**  
**Solution:** The average is  
$\frac{1}{T} f_0^{14} f_0^{14} e^{x+y} dx dy = \frac{1}{14^2} f_0^{14} e^{y} (e^{x}) dy = \frac{1}{14^2} f_0^{14} e^{y} (e^{y} - 1) dy = \frac{1}{14} \left(\frac{e^{2y} - e^{y}}{y}\right)_0^{14} = \frac{1}{14} \left(\frac{e^{28} - e^{14} - \frac{1}{2} + 1}{14}\right) = \frac{1}{196} (e^{28} - 2e^{14} + 1) = \frac{1}{196} (e^{14} - 1)^2$.  

3. (15 points) Find the critical points and classify them as maxima, minima, saddle points, or none of these.

\[ f(x, y) = x^3 + y^3 - 6y^2 - 3x + 9. \]

**Answer:** To find critical points solve for \((x, y)\) where \(f_x = 0\) and \(f_y = 0\)

\[ f_x = 3x^2 - 3 = 0 \implies x = \pm 1 \]
\[ f_y = 3y^2 - 12y = 0 \implies 3y(y - 4) = 0 \implies y = 0 \text{ or } y = 4. \]

Use Second Partial Test, \(d = f_{xx}f_{yy} - f_{xy}^2\), to classify the critical points.

\(f_{xx} = 6x, f_{yy} = 6y - 12, f_{xy} = 0.\)

\((1, 0) \implies d = (6)(-12) = -72 < 0 \implies (1, 0) \text{ is a saddle point}\)
\((-1, 0) \implies d = (-6)(-12) = 72 > 0 \text{ and } f_{xx} < 0 \implies (-1, 0) \text{ is a relative maximum}\)
\((1, 4) \implies d = (6)(12) = 72 > 0 \text{ and } f_{xx} > 0 \implies (1, 4) \text{ is a relative minimum}\)
\((-1, 4) \implies d = (-6)(12) = -72 < 0 \implies (-1, 4) \text{ is a saddle point}\)

4. (15 points) Consider the integral of \(f(x, y) = x\) between \(y = 0\) and \(y = \sqrt{1 - x^2}\).

(i) (5 points) Sketch the region of integration.

(ii) (5 points) Express this integral using Cartesian coordinates and polar coordinates.

**Answer:** Cartesian: \( \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} x \, dy \, dx \), Polar: \( \int_{0}^{\pi} \int_{0}^{1} r^2 \cos \theta \, dr \, d\theta \)

(iii) (5 points) Evaluate the integral.

**Answer:** \( \int_{0}^{\pi} \cos \theta (\frac{r^2}{2}|_{0}^{1}) \, d\theta = \int_{0}^{\pi} \frac{1}{2} \cos \theta \, d\theta = \frac{1}{2} (\sin \theta)|_{0}^{\pi} = 0. \)

5. (15 points) Consider the lamina bounded by the graphs of the equations \(y = 5x^2, y = 0,\) and \(x = 4\) and density \(\rho = kxy.\)

(i) (7 points) Compute the mass of the lamina.

**Answer:**
\[ m = \int_R \rho(x, y) \, dA = \int_{0}^{4} \int_{0}^{5x^2} kxy \, dy \, dx = \int_{0}^{4} kx[(\frac{5x^2}{2})|_{0}^{5x^2}] \, dx = \int_{0}^{4} \frac{25}{2} kx^5 \, dx = \frac{25}{12} kx^6|_{0}^{4} = \frac{25}{12} (4096)k = \frac{204600}{3} k. \]

(ii) (8 points) Determine the center of mass of the lamina.

**Answer:** Center of mass = \((\bar{x}, \bar{y}) = (\frac{M_y}{m}, \frac{M_x}{m})\), where \(M_x = \int_R y \rho(x, y) \, dA, \ M_y = \int_R x \rho(x, y) \, dA\)

\[ M_x = \int_{0}^{4} \int_{0}^{5x^2} kxy^2 \, dy \, dx = \int_{0}^{4} kx[(\frac{5x^2}{2})|_{0}^{5x^2}] \, dx = \int_{0}^{4} \frac{125}{3} kx^7 \, dx = \frac{125}{24} kx^8|_{0}^{4} = \frac{1024000}{3} k \]
\[ M_y = \int_{0}^{4} \int_{0}^{5x^2} kx^2y \, dy \, dx = \int_{0}^{4} kx^2[(\frac{5x^2}{2})|_{0}^{5x^2}] \, dx = \int_{0}^{4} \frac{25}{2} kx^6 \, dx = \frac{25}{12} kx^7|_{0}^{4} = \frac{204600}{3} k \]

\( (\bar{x}, \bar{y}) = (\frac{194400}{179200}, \frac{1024000}{256000}) = (\frac{24}{7}, 40) \)