Homework 4

Exercise 2.1.3
Solution: The principal branch of the logarithm is a function that is analytic on an open set containing \( r \) and whose derivative is \( 1/z \). Since \( r \) is closed, the value of the integral is 0, by the Fundamental Theorem of Calculus for contour integrals, 2.1.7.

Exercise 2.1.12
Solution: We can define a single-valued branch of the function \( \log z \) that is analytic on the set \( C \setminus \{ z \mid \text{Re} \, z \leq 0 \} \). On that region we have \( 1/z = (\log z)' \). The integral is thus 0 by the Fundamental Theorem of Calculus.

Exercise 2.1.15
Solution: Let \( z = x + iy \). Along \( r \), \(|z| = 1\), so

\[
\left| \frac{\sin z}{z^2} \right| = |\sin z| = \frac{1}{2} |e^{ix} - e^{-ix}| = \frac{1}{2} |e^{ix-y} - e^{-ix+y}|
\]

\[
\leq \frac{1}{2}(|e^{ix-y}| + |e^{-ix+y}|) = \frac{1}{2}(|e^{-y}| + |e^y|) \leq \frac{1}{2}(e + e) = e
\]

since \(-1 \leq y \leq 1\) on \( r \). The arc length of \( r \) is \( 2\pi \).

Thus, \( \left| \int_r \frac{\sin z}{z^2} \, dz \right| \leq 2\pi e \).

Exercise 2.2.1
(a) By the Fundamental Theorem of Calculus, \( \int_r (z^3 + 3) \, dz = \left[ \frac{1}{4} z^4 + 3z \right]_1^{-1} = -6 \)

(b) Since \( r \) is a simple closed curve and \( z^3 + 3 \) is entire, \( \int_r (z^3 + 3) \, dz = 0 \) by the Cauchy's Theorem.

(c) Since \( r \) is a simple closed curve and \( e^{\frac{1}{z}} \) is analytic on and inside \( r \), \( \int_r e^{\frac{1}{z}} \, dz = 0 \) by the Cauchy's Theorem.

(d) Since \( r \) is a simple closed curve and \( \cos \left( 3 + \frac{1}{z-3} \right) \) is analytic on and inside \( r \),

\( \int_r \cos \left( 3 + \frac{1}{z-3} \right) \, dz = 0 \) by the Cauchy's Theorem.
Exercise 2.2.9
Solution: First, directly, define \( r(t) = e^{it}, 0 \leq t \leq \pi \). Then \( \int_r \sqrt{z} \, dz = \int_0^\pi \sqrt{e^{it}} \, ie^{it} \, dt = -\frac{2}{3} - \frac{2}{3}i \)

Second, using the Fundamental Theorem of Calculus, \( \int_r \sqrt{z} \, dz = \left[ \frac{2}{3}z^{3/2} \right]_1^{-1} = -\frac{2}{3} - \frac{2}{3}i \)

Exercise 2.2.11
Solution: Using partial fractions, \( \int_r \frac{2x^2 - 15x + 30}{x^3 - 10x^2 + 32x - 32} \, dz = \int_r \frac{2}{x-2} \, dz + \int_r \frac{1}{(x-4)^2} \, dz = 4\pi i + 0 = 4\pi i \)

Exercise 2.3.7
(a) Since the \( 1/z \) is analytic on \( C\setminus\{0\} \) and also the \( r \) and unit circle are homotopic as closed curves in \( C\setminus\{0\} \), the integral is \( 2\pi i \) by the Deformation Theorem.

(b) Note that \( 1/z^2 \) is defined and analytic on \( C\setminus\{0\} \) and is the derivative of \( -1/z \) which is defined and analytic on \( C\setminus\{0\} \). Since the closed curve \( r \) lies in \( C\setminus\{0\} \), the integral is 0 by the Path Independence Theorem.

(c) Since the function \( e^{z \over z} \) is analytic on \( C\setminus\{0\} \) and \( r \) is a closed curve in \( C\setminus\{0\} \) which is homotopic to a point in \( C\setminus\{0\} \). Thus the integral is 0 by the Homotopy Form of Cauchy’s Theorem.

(d) Using partial fractions, \( \int_r \frac{1}{z^2 - 1} \, dz = \frac{1}{2} \left( \int_r \frac{1}{z-1} \, dz - \int_r \frac{1}{z+1} \, dz \right) = \pi i - 0 = \pi i \)

Exercise 2.3.10
(a) Since the function \( \frac{1}{(1-z)^3} \) is analytic on \( C\setminus\{1\} \) and \( r \) is a closed curve in \( C\setminus\{1\} \) which is homotopic to a point in \( C\setminus\{1\} \). Thus the integral is 0 by the Homotopy Form of Cauchy’s Theorem.

(b) Note that \( \frac{1}{(1-z)^3} \) is defined and analytic on \( C\setminus\{1\} \) and is the derivative of \( \frac{1}{2(1-z)^2} \) which is defined and analytic on \( C\setminus\{1\} \). Since the closed curve \( r \) lies in \( C\setminus\{1\} \), the integral is 0 by the Path Independence Theorem.

(c) Since the function \( \frac{1}{(1-z)^3} \) is analytic on \( C\setminus\{1\} \) and \( r \) is a closed curve in \( C\setminus\{1\} \) which is homotopic to a point in \( C\setminus\{1\} \). Thus the integral is 0 by the Homotopy Form of Cauchy’s Theorem.