Exercise 1.1.14
Prove the binomial theorem for complex numbers; that is, letting $z, w$ be complex numbers and $n$ be a positive integer,

$$(z + w)^n = z^n + \binom{n}{1} z^{n-1} w + \binom{n}{2} z^{n-2} w^2 + \cdots + \binom{n}{n} w^n,$$

Where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Use induction on $n$.

Solution:

If $n = 1$, the assertion is true since it just states that

$$z + w = z + w \quad because \quad \binom{1}{1} = 1.$$  

Suppose the assertion is true for $n - 1$, i.e.,

$$(z + w)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} z^{n-1-k} w^k.$$  

Then

$$(z + w)^n = \left(\sum_{k=0}^{n-1} \binom{n-1}{k} z^{n-1-k} w^k\right)(z + w) = \sum_{k=0}^{n-1} \binom{n-1}{k} z^{n-k} w^k + \sum_{k=0}^{n-1} \binom{n-1}{k} z^{n-1-k} w^{k+1}$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} z^{n-k} w^k + \sum_{j=1}^{n-1} \binom{n-1}{j-1} z^{n-j} w^j + w^n$$

$$= z^n + \sum_{k=1}^{n-1} \left(\binom{n-1}{k} + \binom{n-1}{k-1}\right) z^{n-k} w^k + w^n = z^n + \sum_{k=1}^{n} \binom{n}{k} z^{n-k} w^k + w^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k$$

So the assertion holds for $n$. The theorem follows by induction.