Lecture 10 Summary (Chapter 4)

5) Conditional Probability and Independence

(A) Conditional probability

Conditional probability of \( A \) given \( B \):

\[
P(A|B) = \frac{P(AB)}{P(B)} \text{ if } P(B) > 0
\]

Equivalently,

\[
P(AB) = P(A|B)P(B) \text{ if } P(B) \neq 0
\]

and

\[
P(AB) = P(B|A)P(A) \text{ if } P(A) \neq 0.
\]

**Example 12**  Body weight and hypertension

<table>
<thead>
<tr>
<th></th>
<th>Overweight</th>
<th>Normal</th>
<th>Underweight</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hypertensive</td>
<td>0.10</td>
<td>0.08</td>
<td>0.02</td>
<td>0.20</td>
</tr>
<tr>
<td>Not hypertensive</td>
<td>0.15</td>
<td>0.45</td>
<td>0.20</td>
<td>0.80</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0.25</td>
<td>0.53</td>
<td>0.22</td>
<td>1.00</td>
</tr>
</tbody>
</table>

(a) Probability that a person selected at random from this group will have hypertension:

\[ A = \{\text{hypertensive}\} \implies P(A) = 0.2 \]

(b) A person, selected at random from this group is found to be overweight. What is the probability that this person is also hypertensive?

Define \( A = \{\text{hypertensive}\} \) and \( B = \{\text{overweight}\} \). Then

\[
P(A|B) = \frac{P(AB)}{P(B)} = \frac{0.1}{0.25} = 0.4.
\]

- Conditional probability is a probability.

\[
P(\overline{A}|B) = 1 - P(A|B)
\]
(B) **Independence**

A and $B$ are independent if $P(A|B) = P(A)$.

Equivalently, $P(B|A) = P(B)$.

**Example 13** Tossing a fair die:

Let $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3, 4\}$. Then

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{2}{3}$$

$AB = \{2\}$, $AC = \{2, 4\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \neq P(A)$$

$$P(A|C) = \frac{P(AC)}{P(C)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} = P(A)$$

Therefore, $A$ and $B$ are not independent, while $A$ and $C$ are independent.

$A$ and $B$ are independent if and only if $P(AB) = P(A)P(B)$. Also,

$$P(AB) = P(A)P(B) \quad \text{and} \quad P(\bar{A}\bar{B}) = P(\bar{A})P(\bar{B})$$

**Example 13 (continued)**

$$P(AB) = \frac{1}{6} \neq P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Thus, $A$ and $B$ are not independent.

$$P(AC) = \frac{1}{3} = P(A)P(C) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$$

Therefore, $A$ and $C$ are independent.