Optimized preparation of quantum states by conditional measurements

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We introduce a general strategy for preparation of arbitrary quantum states via optimal control of repeated conditional measurements. The effectiveness of this strategy in generating finite Fock-state superpositions with a high level of confidence from experimentally accessible coherent states is demonstrated for the simple and well-known Jaynes-Cummings model dynamics. [S1050-2947(96)07406-9]

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State preparation of quantum systems is a prerequisite for studying fundamental aspects of quantum measurement theory [1], as well as for encoding quantum information [2] and its processing (computing) [3]. Problems of state preparation have been dealt with most extensively in the realm of cavity quantum electrodynamics [4–8]. One proposal for state preparation of a cavity field mode [8] relies entirely on its unitary evolution, via coupling with a rather complex system, an atom having several Zeeman sublevels. This coupling results (under perfectly adiabatic conditions) in a one-to-one mapping of the initial sublevel superposition to a superposition of Fock (photon-number) states. Can one alternatively use a simple field-atom interaction, e.g., the resonant Jaynes-Cummings (JC) model [4] or the off-resonant Kerr-like interaction [5], followed by a measurement on the atom leaving the cavity, and repeat the process over and over again until the desired state of the field is attained?

In general, measurements of atomic observables after the interaction would yield random results for the prepared field state [4,5]. In order to prepare predetermined field states, the conditional measurement (CM) approach has been suggested [6]. In this approach, only those sequences of atoms in which each atom is found after the interaction to be in a chosen state are used to guide the field evolution to the desired state, whereas all other measurement sequences are discarded, at the price of atomic post selection probability, which is less than unity at each step of the sequence. The CM approach has been significantly enriched by a recipe for constructing an arbitrary superposition of Fock states [7]. It is based on a recurrence relation, which allows one to retract the desired superposition back to the starting vacuum state, by determining the possible initial atomic states and interaction times (in the JC model) at each step of the CM sequence. The practical restriction on this recipe is that the probability of the resulting CM sequences falls off rapidly with the maximal photon number in the superposition.

Our aim here is to address the basic questions of state preparation via quantum measurements: (a) Given a simple field-atom interaction, as in the JC model, and a choice of experimentally realizable initial field states (e.g., coherent states), can the field converge to any desired “target” state to within the required accuracy via a finite number of measurements? Hilbert space topology arguments supported by numerical calculations are given to show that such convergence is in general attained, provided that the number of control parameters per CM is comparable to the dimensionality of the target-state subspace. (b) How can one choose a CM sequence connecting the initial and “target” states, so as to maximize its success probability and minimize its length (the required number of CMs)? Although in principle it should be possible to choose the CM sequence with the highest ratio of probability to length, in practice such optimization amounts to the formidable task of a global search over a huge parameter space (whose dimensionality is the number of parameters per CM times the maximal admissible number of CMs). We demonstrate that there is a simple and computationally fast alternative, namely, stepwise optimization by search over the parameter space of one CM at a time, allowing one to choose a high-probability CM sequence from among those that converge to the target state monotonically.

Let us first formulate our strategy in general terms, suitable for any dynamical model. Suppose that we have started from the field state \( |\psi_0\rangle = \sum_n C_n |n\rangle \). After \( K-1 \) CMs the field state is \( |\psi_{K-1}\rangle = \sum_n C_n |n\rangle \), and we are trying to obtain the “target” state \( |\psi_f\rangle = \sum_{n = n_{\min}}^{n_{\max}} C_n |n\rangle \). This choice of initial and final atomic states and of \( \tau_K \) corresponds to choosing a field-state CM transformation

\[
|\psi_{K-1}\rangle \rightarrow |\psi_f\rangle = P_K^{-1/2} |\phi^{(f)}_K\rangle U(\tau_K) |\psi_{K-1}\rangle \otimes |\phi^{(i)}_K\rangle,
\]

where \( P_K = \sum_{n = n_{\min}}^{n_{\max}} \langle |\psi_{K-1}\rangle |\psi_{K-1}\rangle |\phi^{(i)}_K\rangle |\phi^{(i)}_K\rangle \) is the success probability of the corresponding CM.

The application of a sequence of CM transformations to an initial coherent state effects convergence to the target state via two processes taking place in parallel: (a) gradual elimination (filtering out) of all undesired Fock states \( |n\rangle \) from the starting coherent state, while leaving the amplitudes...
of the Fock states \( n_{\text{min}} \leq n \leq n_{\text{max}} \) almost intact—this is achievable by adjusting the control parameters so as to leave the CM transformation near unity for \( n_{\text{min}} \leq n \leq n_{\text{max}} \) and considerably less than unity for all other \( n \) (see below); (b) amplitude fine tuning in the Hilbert subspace spanned by Fock states \( |n\rangle \), \( n_{\text{min}} \leq n \leq n_{\text{max}} \), wherein the target state is contained—after the first few CMs the field state remains confined to this subspace by the filtering action of the first process.

The physical states within the above \((N = n_{\text{max}} - n_{\text{min}} + 1)\)-dimensional Hilbert subspace are represented by unit norm vectors with global phase factors ignored. They constitute a manifold of \( D_{N} = 2(N - 1) \) real dimensions. An important part of the optimization is to fix a distance function on this manifold, in terms of which proximity to the target site is measured. A particularly useful family of such distances

\[
d_{n,K} = \left\{ a d_{\text{mod,K}}^2 + d_{\text{ph,K}}^2 \right\}^{1/2}
\]

is obtained by assigning an adjustable weight \( a \) to the moduli distance from the target

\[
d_{\text{mod,K}} = \left\{ \sum_n \left| \langle C_{n,K} |^2 - |C_{n,l}|^2 \right|^2 \right\}^{1/2}
\]

relative to the phase-factor distance

\[
d_{\text{ph,K}} = \left\{ \sum_n \left| e^{i\chi_n} C_{n,K} / |C_{n,l}| - C_{n,l} / |C_{n,l}|^2 \right|^2 \right\}^{1/2}.
\]

The phase factor \( e^{i\chi_n} \) is chosen to be such that \( \chi_n + \arg(C_{n,K}) = \arg(C_{n,l}) \) for a chosen \( n \), thus eliminating the arbitrariness in the overall phases of \( |\psi_K\rangle \) and \( |\psi_l\rangle \). The current field state \( |\psi_K\rangle \) can then be viewed as lying on the \((D_N - 1)\)-dimensional hypersurface of states equidistant (in terms of \( d_n \)) from the target state [Fig. 1(a)]. The family of all possible CM transformations acting on the field state includes transformations close to the identity, whence it spans a submanifold containing \( |\psi_K\rangle \), whose real dimensionality is, normally, the number of control parameters \( D_c \) [Fig. 1(a)].

Assuming that \( D_c = D_N \), how can we achieve optimized convergence to the target? The answer is that, generically, the intersection of the spanned submanifold with the equidistant-state hypersurface, which occurs at \( |\psi_K\rangle \), is transversal for any \( D_c \geq 1 \). Hence the spanned submanifold contains states closer to the target than \( |\psi_K\rangle \), as well as states further away from it. Among those states closer to the target than \( |\psi_K\rangle \), we may choose an optimal one, having high success probability \( P_{K+1} \) for the corresponding CM transformation. The chosen \( |\psi_{K+1}\rangle \) minimizes a cost function \( G_{K+1} \), which, albeit arbitrary, must increase with \( d_{n,K+1} \) and be reduced with success probability \( P_{K+1} \). The simplest choice, which has proven to be highly effective is

\[
G_{K+1} = \frac{d_{n,K+1}^2}{P_{K+1}^2}
\]

Here the adjustable exponent \( r \geq 0 \) determines the sensitivity to success probability: The maximization of \( P_{K+1} \) is most drastic when \( r \) is large, but excessive \( r \) values render \( G_{K+1} \) insensitive to distance, thus hampering convergence.

Repeating this process, again and again, we obtain a sequence of field states with monotonically decreasing distances to the target along with maximized probabilities. This holds true for arbitrary \( D_N \) and \( D_c \geq 1 \), but as \( D_c \) increases, so does the choice of optimized states \( |\psi_K\rangle \), allowing higher CM success probabilities. Moreover, when \( D_c \) approaches \( D_N \) we may expect the residual distance \( \lim_{K \to \infty} d_{n,K} \) to be small, allowing good proximity to the target.

We shall now apply this general optimization strategy to experimental setups which realize the resonant JC model [4], where convergence is successful for \( N \leq 4 \) \((D_N \leq 6)\) since \( D_c = 5 \) (see below). For convenience, the quantum states of the cavity field and atomic systems are written in the interaction picture.

(1) The first stage of the considered setup is the atomic preparation region: each atom enters the region in its ground state \( |g\rangle \) and with controllable velocity, then interacts with classical fields which transform \( |g\rangle \) into a superposition of excited and ground states with two independent parameters

\[
|\psi_{i,K}\rangle = \alpha_{K} |e\rangle + \beta_{K} |g\rangle,
\]

the superscript signifying the initial atomic state.

(2) The second stage of the setup is the cavity, in which the atom interacts for a time \( t_K \) (determined by its velocity) with the resonant field mode. The starting field state \( |\psi_0\rangle = \sum_n e^{i\chi_n} |n\rangle \) is a coherent (or squeezed) state prepared by means of a classical oscillator that is phase locked (synchronized) with the resonant classical field in stage 1. The second stage is described by the unitary evolution operator of the resonant JC model, \( U(t_K) \), which entangles the field and atom states according to

\[
|n\rangle|e\rangle \rightarrow \cos \theta_{n,K} |n\rangle|e\rangle - i \sin \theta_{n,K} |n + 1\rangle|g\rangle,
\]

\[
|n\rangle|g\rangle \rightarrow \cos \theta_{n-1,K} |n\rangle|g\rangle - i \sin \theta_{n-1,K} |n - 1\rangle|e\rangle,
\]

where \( \theta_{n} = \frac{\pi}{2} n / \sqrt{n + 1} \), \( 2 \lambda \) being the vacuum Rabi frequency (and \( |\downarrow\rangle \) is formally defined to be zero).

(3) Upon exiting the cavity, the atom encounters the last stage, in which it interacts again with classical fields. This last stage is aimed at projecting the entangled field-atom state onto the atomic state

\[
|\psi_{f,K}\rangle = \alpha_{f,K} |e\rangle + \beta_{f,K} |g\rangle.
\]

Here \( \alpha_{f,K} \) and \( \beta_{f,K} \) are the final (postselected) counterparts of the initial atomic parameters in Eq. (6). The classical fields in this stage are so set as to unitarily transform the state \( |\psi_{f,K}\rangle \) into \( |g\rangle \), whence successful detection of the atoms in the state \( |g\rangle \) then corresponds to projection onto \( |\psi_{f,K}\rangle \), whereas detection of the \( |e\rangle \) state corresponds to projection onto the state orthogonal to \( |\psi_{f,K}\rangle \).

The three consecutive stages described above: preparation of \( |\psi_{i,K}\rangle \), then resonant JC unitary evolution \( U(t_{K}) \), followed by projection onto \( |\psi_{f,K}\rangle \), effect the field-state transformation

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where \( \theta_{n} = \frac{\pi}{2} n / \sqrt{n + 1} \), \( 2 \lambda \) being the vacuum Rabi frequency (and \( |\downarrow\rangle \) is formally defined to be zero).

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The filtering out of unwanted parts of the initial Fock-state distribution.
From numerous computations based on the present strategy, we can infer that the success probability of the optimized reshaping process of optimization, lies on the phase redundancy, as well as $\lambda \tau_K$. The accuracy to which the target state can be attained is (at best) that of controlling the above five parameters.

From numerous computations based on the present strategy we can infer that the success probability of the optimized sequence is close to, and often higher than the squared projection $|\langle \psi_{t} | \psi_{t} \rangle|^2$ of the starting coherent state on the target state. This implies that convergence to the target state is genuine optimization, and not merely the filtering-out (erasure) of unwanted parts of the initial Fock-state distribution. The filtering out is effected by the diagonal terms $(C_{n,K-1})$ in (9), for which $\lambda \tau_K$ is such that $|\cos \theta_{n,K}| \ll 1$, whereas for $n_{\min} \leq n \leq n_{\max}$, $|\cos \theta_{n,K}| \approx 1$. By contrast, the off-diagonal terms $(C_{n+1,K-1})$ in (9) are responsible for the fine tuning (reshaping) of the Fock-state distribution.

The results of the optimization strategy outlined in Eqs. (5)–(9) are illustrated in Figs. 1 and 2. In Fig. 1 we explore...
the difficult challenge of starting from a coherent state and attaining a superposition of three successive number states whose amplitudes have prescribed different phases, as well as, moduli. Such a state can encode one quantum ternary digit. The optimization yields a CM sequence whose success probability $P_{\text{seq}}$ is twice the squared projection of the initial state on the target state, $P_{\text{seq}} = 2|\langle \psi_0 | \psi_t \rangle|^2$ (8% compared to 4% initially). In Fig. 2 we optimize the generation of a superposition of four successive number states with different moduli but equal phases of the amplitudes. Such a state (with different phases allowed) can encode two quantum binary digits (qubits) [3]. By optimization we obtain for this state $P_{\text{seq}} = |\langle \psi_0 | \psi_t \rangle|^2$. Typically, five to ten CMs suffice to attain 95% and higher overlap with the target.

The outlined strategy for field-state preparation via optimal control of repeated CMs is universal, in that it is applicable to any field-atom interaction and starting-field state (as long as it is pure). The effectiveness of this strategy in generating finite Fock-state superpositions, which has been demonstrated in Figs. 1, and 2, is contingent on two conditions: (a) The lifetime (decoherence time) of the mode divided by the photon number must be much longer than the state preparation time. This is achievable at present in high-$Q$ microcavities [4,6]. (b) High detection efficiency of the final atomic states (90–95%, according to our numerical checks) is needed for high reliability of state preparation.

If these two conditions are satisfied, then the present CM strategy allows one to “load” the cavity field with quantum information (m qubits in superpositions of $2^m$ Fock states) at a rather high rate: $M$ repetitions of the CM sequence, whose size $K_{\text{max}}$ suffices for convergence to the target state (within the accuracy margin), can guarantee the state “loading” with confidence level $P_{\text{con}}$ (desired success probability), if

$$M = \log(1 - P_{\text{con}})/\log(1 - P_{\text{seq}}).$$

where $P_{\text{seq}}$ is the success probability of a single CM sequence. For $P_{\text{con}} = 0.95$ and $P_{\text{seq}} = 0.05$ this yields $M = 58$, whence a rather modest number of trials; $MK_{\text{max}} \approx 500$ is then required for $K_{\text{max}} \approx 10$. Practically, the loading rate $1/M(K_{\text{max}}/\tau_{\text{reset}})$ is determined by the average interaction time $\tau$, the time to reset the field to its initial state after an unsuccessful measurement sequence, $\tau_{\text{reset}}$, and the requirement that $P_{\text{con}}$ be comparable with the atomic detection efficiency or the field-decoherence probability (during the state preparation). By comparison, $P_{\text{con}}$ in unitary nearly adiabatic
schemes [8] is limited by the probability of intermediate-level decay or field decoherence, whereas the loading rate is determined by the duration of the multilevel atomic-state preparation and the subsequent nearly adiabatic coherence transfer to the field. These estimates suggest that our optimized CM strategy, based on simple dynamics, such as the JC model, is a viable alternative to unitary evolution schemes involving multilevel-atom dynamics.