Weighted ENO Schemes

Xiaolei Chen
Advisor: Prof. Xiaolin Li

Department of Applied Mathematics and Statistics
Stony Brook University, The State University of New York

February 7, 2014
1. Background

2. WENO Schemes
   - Basic Idea
   - Choice of the Weights
   - Flux Splitting

3. Modified Weights
   - Mapped WENO Schemes
   - WENO-Z Scheme
Background

1D Scalar Hyperbolic Equation:

\[ u_t + (f(u))_x = 0 \]

Assume the grid is uniform and solve the hyperbolic equation directly using a conservative approximation to the spatial derivative.

\[ \frac{du_i(t)}{dt} = -\frac{1}{\Delta x} (\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}}), \]

where \( u_i(t) \) is the numerical approximation to the point value \( u(x_i, t) \), and \( \hat{f}_{i+\frac{1}{2}} \) is called numerical flux.

Question: How to approximate the numerical flux \( \hat{f}_{i+\frac{1}{2}} \)?
Basic Idea

Figure: Stencil of fifth order WENO scheme
At smooth region, each sub-stencil gives third (k-th) order numerical fluxes \( v_{i+\frac{1}{2}}^- \) and \( v_{i-\frac{1}{2}}^+ \).

For example, on stencil \( S_1 \),

\[
\begin{align*}
 v_{i+\frac{1}{2}}^{(1)-} &= -\frac{1}{6} \bar{v}_{i-1} + \frac{5}{6} \bar{v}_i + \frac{1}{3} \bar{v}_{i+1} \\
v_{i-\frac{1}{2}}^{(1)+} &= \frac{1}{3} \bar{v}_{i-1} + \frac{5}{6} \bar{v}_i - \frac{1}{6} \bar{v}_{i+1}
\end{align*}
\]

Similarly, we have

- \( v_{i+\frac{1}{2}}^{(0)-} \) and \( v_{i-\frac{1}{2}}^{(0)+} \) on stencil \( S_0 \)
- \( v_{i+\frac{1}{2}}^{(2)-} \) and \( v_{i-\frac{1}{2}}^{(2)+} \) on stencil \( S_2 \)

*The coefficients come from reconstruction process.*
Basic Idea

Apply a weight to each stencil and a fifth \((2k-1)\)-th order WENO scheme is obtained. Assume the weights are \(w_0, w_1, w_2\). Then, we require

\[
\begin{align*}
    w_r &\geq 0, \\
    \sum_{s=0}^{k-1} w_s &= 1,
\end{align*}
\]

for stability and consistency.

The fifth order fluxes are given by

\[
\begin{align*}
    v_{i+\frac{1}{2}}^- &= \sum_{r=0}^{k-1} w_r v_{i+\frac{1}{2}}^{(r)-}, \\
    v_{i-\frac{1}{2}}^+ &= \sum_{r=0}^{k-1} w_r v_{i-\frac{1}{2}}^{(r)+},
\end{align*}
\]

Question: How to choose the weights?
Choice of the Weights

If the function $v(x)$ is smooth in all of the candidate stencils, there are constants $d_r$ such that

$$v_{i+\frac{1}{2}}^− = \sum_{r=0}^{k-1} d_r v_{i+\frac{1}{2}}^{(r)-} = v(x_{i+\frac{1}{2}}) + O(\Delta x^{2k-1}),$$

$$v_{i-\frac{1}{2}}^+ = \sum_{r=0}^{k-1} \tilde{d}_r v_{i-\frac{1}{2}}^{(r)+} = v(x_{i-\frac{1}{2}}) + O(\Delta x^{2k-1}).$$

For example, when $k=3$,

$$d_0 = \frac{3}{10}, \quad d_1 = \frac{3}{5}, \quad d_2 = \frac{1}{10},$$

$$\tilde{d}_0 = \frac{1}{10}, \quad \tilde{d}_1 = \frac{3}{5}, \quad \tilde{d}_2 = \frac{3}{10}. $$
Choice of the Weights

In this smooth case, we would like to have

$$w_r = d_r + O(\Delta x^{k-1}).$$

Form of the Weights:

$$w_r = \frac{\alpha_r}{\sum_{s=0}^{k-1} \alpha_s}, \ r = 0, \cdots, \ k - 1$$

$$\alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2}$$

where $\beta_r$ are the so-called "smooth indicators" of the stencil $S_r$. We require

- if $v(x)$ is smooth in the stencil $S_r$, then $\beta_r = O(\Delta x^2)$
- if $v(x)$ has a discontinuity inside the stencil $S_r$, then $\beta_r = O(1)$
Smooth Indicators

Let the reconstruction polynomial on the stencil $S_r$ be denoted by $p_r(x)$. Then, define

$$
\beta_r = \sum_{l=1}^{k-1} \int_{x_i-\frac{1}{2}}^{x_i+\frac{1}{2}} \Delta x^{2l-1} \left( \frac{d^l p_r(x)}{dx^l} \right)^2 dx
$$

When $k=3$,

$$
\beta_0 = \frac{13}{12} (\bar{v}_i - 2\bar{v}_{i+1} + \bar{v}_{i+2})^2 + \frac{1}{4} (3\bar{v}_i - 4\bar{v}_{i+1} + \bar{v}_{i+2})^2,
$$

$$
\beta_1 = \frac{13}{12} (\bar{v}_{i-1} - 2\bar{v}_i + \bar{v}_{i+1})^2 + \frac{1}{4} (\bar{v}_{i-1} - \bar{v}_{i+1})^2,
$$

$$
\beta_2 = \frac{13}{12} (\bar{v}_{i-2} - 2\bar{v}_{i-1} + \bar{v}_i)^2 + \frac{1}{4} (\bar{v}_{i-2} - 4\bar{v}_{i-1} + 3\bar{v}_i)^2
$$
Flux Splitting

Lax-Friedrichs Splitting:

\[ f^\pm(u) = \frac{1}{2}(f(u) \pm \alpha u), \]

where \( \alpha = \max_u |f'(u)| \) over the relevant range of \( u \).

FD WENO Procedure with Flux Splitting:

1. Identify \( \bar{v}_i = f^+(u_i) \) and use WENO procedure to obtain \( v^-_{i+\frac{1}{2}} \), and take \( \hat{f}^+_{i+\frac{1}{2}} = v^-_{i+\frac{1}{2}} \).
2. Identify \( \bar{v}_i = f^-(u_i) \) and use WENO procedure to obtain \( v^+_{i+\frac{1}{2}} \), and take \( \hat{f}^-_{i+\frac{1}{2}} = v^+_{i+\frac{1}{2}} \).
3. Form the numerical flux as \( \hat{f}_{i+\frac{1}{2}} = \hat{f}^+_{i+\frac{1}{2}} + \hat{f}^-_{i+\frac{1}{2}} \).
Mapped WENO Schemes

\begin{align*}
g_r(w) &= \frac{w(d_r + d_r^2 - 3d_r w + w^2)}{d_r^2 + w(1 - 2d_r)} \\
\alpha_r^* &= g_r(w_r^{JS}) \\
w_r^M &= \frac{\alpha_r^*}{\sum_{s=0}^{2} \alpha_s^*}
\end{align*}

Advantage: fifth order accuracy at critical points
Disadvantage: the weight of the stencil that contains discontinuity becomes larger after the map; more computational cost.

Xiaolei Chen
Advisor: Prof. Xiaolin Li

Background
WENO Schemes
Modified Weights

Mapped WENO Schemes
WENO-Z Scheme

Weighted ENO Schemes
WENO-Z Scheme

\[ \tau_5 = |\beta_0 - \beta_2| \]

\[ \alpha_r^z = d_r (1 + \left( \frac{\tau_5}{\beta_k + \epsilon} \right)^q) \]

\[ w_r^z = \frac{\alpha_r^z}{\sum_{s=0}^{2} \alpha_s^z} \]

Advantage: fifth order accuracy at critical points with q=2; computationally more efficient than mapped WENO.
