Boundary Conditions in Projection Methods

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General Introduction

Incompressible Navier-Stokes Equation

In an $n$-dimensional bounded domain $\Omega$, consider the incompressible Navier-Stokes equations

$$
\rho(u_t + (u \cdot \nabla)u) = -\nabla p + \nu \nabla^2 u
$$

$$
\nabla \cdot u = 0
$$

$$
B(u) = 0
$$

where $u$ is the fluid velocity, $p$ is the pressure, $\rho$ and $\nu$ are the density and viscosity of the fluid respectively.
Projection Method

The second-order time-discrete semi-implicit scheme

\[ \frac{u^{n+1} - u^n}{dt} + \nabla p = -[(u \cdot \nabla)u]^{n+\frac{1}{2}} + \frac{\nu}{2} \nabla^2 (u^{n+1} + u^n) \]

\[ \nabla \cdot u^{n+1} = 0 \]

\[ B(u^{n+1}) = 0 \]

Projection method: fractional step
Projection Method (cont.)

- compute the intermediate velocity
  \[ \frac{u^* - u^n}{dt} + \nabla q = -[(u \cdot \nabla)u]^n + \frac{v}{2} \nabla^2 (u^* + u^n) \]
- perform the projection
  \[ u^* = u^{n+1} + dt \nabla \phi^{n+1} \]
  \[ \nabla \cdot u^{n+1} = 0 \]
- update the pressure
  \[ p^{n+\frac{1}{2}} = q + L(\phi^{n+1}) \]

where \( q \) is an approximation of the pressure \( p^{n+\frac{1}{2}} \).
In Step 1, boundary conditions for the intermediate velocity is required.

- If \( q = p^{n-\frac{1}{2}} \), then
  \[ B(u^*) = B(u^{n+1}) \]
gives second-order accuracy in velocity.

- If \( q = 0 \), then
  \[ B(u^*) = B(u^{n+1}) \]
only gives first-order accuracy in velocity. But,
  \[ B(u^*) = B(u^{n+1} + \nabla \phi^n) \]
gives second-order accuracy in velocity.
Boundary Conditions

In Step 2, an elliptic equation $\nabla^2 \phi = \nabla \cdot u^*/dt$ is solved.

- If $q = p^{n-\frac{1}{2}}$, then
  \[ B(\nabla \phi^{n+1}) = 0, \quad p^{n+\frac{1}{2}} = q - \frac{\nu dt}{2} \nabla^2 \phi^{n+1} \]
  gives second-order accuracy in velocity.

- If $q = 0$, then
  \[ B(\nabla \phi^{n+1}) = 0, \quad p^{n+\frac{1}{2}} = \phi^{n+1} - \frac{\nu dt}{2} \nabla^2 \phi^{n+1} \]
  only gives first-order accuracy in velocity. But,
  \[ B(\nabla \phi^{n+1}) = B(u^* - u^{n+1}), \quad p^{n+\frac{1}{2}} = \phi^{n+1} - \frac{\nu dt}{2} \nabla^2 \phi^{n+1} \]
  gives second-order accuracy in velocity.
Boundary Conditions (cont.)

- **Question 1:** how to calculate $\nabla \cdot \mathbf{u}^*$ near the irregular rigid body boundary (e.g. a circle).

  $$
  \nabla \cdot \mathbf{u}^*_{i,j} = \frac{u_{i+1,j}^* - u_{i-1,j}^*}{2 \, dx} + \frac{v_{i,j+1}^* - v_{i,j-1}^*}{2 \, dy}
  $$

- **Question 2:** if there is irregular rigid body boundary (e.g. a circle), how to discretize the B.C. $\nabla \phi^{n+1}$.

  $\nabla \phi^{n+1}_{i,j} = \frac{\phi_{i+1,j}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i-1,j}^{n+1}}{dx^2} + \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^{n+1}}{dy^2} = \frac{\nabla \cdot \mathbf{u}_{i,j}^*}{dt}$
Boundary Conditions (cont.)
Reflective (Mirror) B.C.

Idea: If a neighbor point is outside the boundary,

- Step 1: find the normal vector at the boundary
- Step 2: according to the normal vector, locate the mirror point if the neighbor
- Step 3: interpolate the velocity/pressure at the mirror point by the interior points around it
- Step 4: consider the neighbor point and its mirror point have the same state.
Boundary Conditions (cont.)

Reflective (Mirror) B.C.

- For question 1, since all the intermediate velocity of the interior points are known, it is straightforward to deal with the boundary.

- For question 2, because the pressure of the interior points are unknown, we cannot directly interpolate the value. The weights of each around point which is used to do the interpolation are added to the coefficient matrix.
Future Work

- The implementation of the idea.
- Check the divergence around the boundary.
- Look for other ways to deal with the B.C.s.
- Extend the idea to movable body boundary,
Reference

