Rigid Body Rotation

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Introduction

- **Question:** How many degrees of freedom in the rigid body motion?
- **Answer:** In total, there are 6 degrees of freedom.
  1) 3 for the translational motion
  2) 3 for the rotational motion
- **Translational Motion:** center of mass velocity
- **Rotational Motion:** angular velocity, euler angles, euler parameters
Angular Velocity

In physics, the angular velocity is defined as the rate of change of angular displacement.

In 2D cases, the angular velocity is a scalar.

Equation: \( \frac{d\theta}{dt} = w \)

Linear velocity: \( v = w \times r \)
Angular Velocity

In the 3D rigid body rotation, the angular velocity is a vector, and use $\mathbf{w} = (w_1, w_2, w_3)$ to denote it.
Angular Momentum

- In physics, the angular momentum ($L$) is a measure of the amount of rotation an object has, taking into account its mass, shape and speed.

- The $L$ of a particle about a given origin in the body axes is given by

$$ L = r \times p = r \times (mv) $$

where $p$ is the linear momentum, and $v$ is the linear velocity.

- Linear velocity: $v = w \times r$

$$ L = mr \times (w \times r) = m(wr^2 - r(r \cdot w)) $$
Angular Momentum

Each component of the angular Momentum is

\[ L_1 = m(r^2 - x^2)w_1 - mxyw_2 - mxzw_3 \]
\[ L_2 = -myxw_1 + m(r^2 - y^2)w_2 - myzw_3 \]
\[ L_3 = -mzxw_1 - mzyw_2 + m(r^2 - z^2)w_3 \]

It can be written in the tensor formula.

\[ L = lw \]
\[ I = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}, I_{ab} = m(r^2 \delta_{ab} - ab) \]
Inertia Tensor

- Moment of inertia is the mass property of a rigid body that determines the torque needed for a desired angular acceleration about an axis of rotation. It can be defined by the Inertia Tensor ($I$), which consists of nine components ($3 \times 3$ matrix).

- In continuous cases,

$$I_{ab} = \int_V \rho(V)(r^2 \delta_{ab} - ab) dV$$

- The inertia tensor is constant in the body axes.
Inertia Tensor

\[ I = \begin{bmatrix} \frac{2}{5}mr^2 & 0 & 0 \\ 0 & \frac{2}{5}mr^2 & 0 \\ 0 & 0 & \frac{2}{5}mr^2 \end{bmatrix} \]

\[ I = \begin{bmatrix} \frac{3}{5}mh^2 + \frac{3}{20}mr^2 & 0 & 0 \\ 0 & \frac{3}{5}mh^2 + \frac{3}{20}mr^2 & 0 \\ 0 & 0 & \frac{3}{10}mr^2 \end{bmatrix} \]

\[ I = \begin{bmatrix} \frac{1}{12}m(3r^2 + h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2 + h^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix} \]
Euler’s Equations of Motion

- Apply the eigen-decomposition to the inertial tensor and obtain

\[ I = Q\Lambda Q^T, \Lambda = \text{diag}\{I_1, I_2, I_3\} \]

where \( I_i \) are called principle moments of inertia, and the columns of \( Q \) are called principle axes. The rigid body rotates in angular velocity \((w_x, w_y, w_z)\) with respect to the principle axes.

- Specially, consider the cases where \( I \) is diagonal.

\[ I = \text{diag}\{I_1, I_2, I_3\}, \quad L_i = I_i w_i, i = 1,2,3 \]

The principle axes are the same as the \( x, y, z \) axes in the body axes.
Euler’s Equations of Motion

The time rate of change of a vector $G$ in the space axes can be written as

$$\left(\frac{dG}{dt}\right)_s = \left(\frac{dG}{dt}\right)_b + w \times G$$

where $\left(\frac{dG}{dt}\right)_b$ is the time rate of change of the vector $G$ in the body axes.

Apply the formula to the angular momentum $L$

$$\left(\frac{dL}{dt}\right)_s = \left(\frac{dL}{dt}\right)_b + w \times G$$
Euler’s Equations of Motion

- Use the relationship $L = Iw$ to obtain

$$\frac{d(Iw)}{dt} + w \times L = \left(\frac{dL}{dt}\right)_s = \tau$$

The second equality is based on the definition of torque.

$$\frac{dL}{dt} = \frac{d(r \times p)}{dt} = r \times \frac{dp}{dt} = r \times F = \tau$$

- Euler’s equation of motion

$$\begin{align*}
I_1\dot{w}_1 - w_2w_3(I_2 - I_3) &= \tau_1 \\
I_2\dot{w}_2 - w_3w_1(I_3 - I_1) &= \tau_2 \\
I_3\dot{w}_3 - w_1w_2(I_1 - I_2) &= \tau_3
\end{align*}$$
Euler Angles

- The Euler angles are three angles introduced to describe the orientation of a rigid body.
- Usually, use \( \phi, \theta, \psi \) to denote them.
Euler Angles

The order of the rotation steps matters.
Euler Angles

Three rotation matrices are

\[ D = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \]

\[ B = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Euler Angles

Then, the rotation matrix is

\[ A = BCD = \begin{pmatrix}
\cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\
-sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\
\sin\theta\sin\phi & \sin\theta\cos\phi & \cos\theta
\end{pmatrix} \]

This matrix can be used to convert the coordinates in the space axes to the coordinates in the body axes.

The inverse is \( A^{-1} = A^T \).

Assume \( x \) is the coordinates in the space axes, and \( x' \) is the coordinates in the body axes.

\[ x' = Ax \]
\[ x = A^T x' \]
Euler Angles

The relationship between the angular velocity and the euler angles is

\[
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\dot{\phi}
\end{pmatrix} + D^T \begin{pmatrix}
\dot{\theta} \\
0
\end{pmatrix} + D^T C^T \begin{pmatrix}
0 \\
0 \\
\dot{\psi}
\end{pmatrix} = J^{-1} \begin{pmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix}
\]

\[
J^{-1} =
\begin{pmatrix}
0 & \cos \phi & \sin \theta \sin \phi \\
0 & \sin \phi & -\sin \theta \cos \phi \\
1 & 0 & \cos \theta
\end{pmatrix}
\]

\[
\begin{pmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{pmatrix} =
\begin{pmatrix}
-\sin \phi \cot \theta & \cos \phi \cot \theta & 1 \\
\cos \phi & \sin \phi & 0 \\
\sin \phi \csc \theta & -\cos \phi \csc \theta & 0
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}
\]
Euler Parameters

The four parameters $e_0, e_1, e_2, e_3$ describe a finite about an arbitrary axis. The parameters are defined as

Def: $e_0 = \cos \frac{\varphi}{2}$

And $(e_1, e_2, e_3) = \vec{n}\sin(\frac{\varphi}{2})$
Euler Parameters

In this representation, $\vec{n}$ is the unit normal vector, and $\varphi$ is the rotation angle.

Constraint: $e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$

$$A = \begin{pmatrix}
e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\
2(e_1e_2 - e_0e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2e_3 + e_0e_1) \\
2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 - e_1^2 + e_2^2 - e_3^2
\end{pmatrix}$$
Euler Parameters

The relationship between the angular velocity and the euler parameters is

\[
\begin{pmatrix}
\dot{e}_0 \\
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3 \\
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
-e_1 & -e_2 & -e_3 \\
e_0 & e_3 & -e_2 \\
e_2 & -e_1 & e_0 \\
e_2 & -e_1 & e_0 \\
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
w_3 \\
\end{pmatrix}
\]

The relationship between the euler angles and the euler parameters is

\[
e_0 = \cos \left( \frac{\phi + \psi}{2} \right) \cos \left( \frac{\theta}{2} \right),
\quad e_1 = \cos \left( \frac{\phi - \psi}{2} \right) \sin \left( \frac{\theta}{2} \right)
\]

\[
e_2 = \sin \left( \frac{\phi - \psi}{2} \right) \sin \left( \frac{\theta}{2} \right),
\quad e_3 = \sin \left( \frac{\phi + \psi}{2} \right) \cos \left( \frac{\theta}{2} \right)
\]
Numerical Simulations

- **Question:** How to propagate the rigid body?
- **Answer:** There are three ways. But not all of them work good in numerical simulation.
  1) Linear velocity
  2) Euler Angles
  3) Euler Parameters
Numerical Simulations

- Propagate by linear velocity
  \[ \frac{dr}{dt} = v = w \times r \]
  \[ r^n \rightarrow r^{n+1} \]

Directly use the last step location information.

- Propagate by euler angles or euler parameter
  1) Implemented by rotation matrix.
  2) The order of propagate matters.
  3) e.g. \( \Delta A \cdot A = \Delta B \Delta C \Delta D \cdot BCD \neq (\Delta B \cdot B)(\Delta C \cdot C)(\Delta D \cdot D) \)

Inverse back to the initial location by the old parameters and propagate by the new parameters.
Numerical Simulations

- Update angular velocity by the Euler’s equation of motion with 4-th order Runge-Kutta method.
- Initial angular velocity: $w_1 = 1, w_2 = 0, w_3 = 1$
- Principle Moment of Inertia:
  \[ I_1 = I_2 = 0.6375, I_3 = 0.075 \]
- Analytic Solution
  \[ w_1 = \cos\left(\frac{I_3 - I_1}{I_1} w_3 t\right) \]
  \[ w_2 = \sin\left(\frac{I_3 - I_1}{I_1} w_3 t\right) \]
Numerical Simulations
Numerical Simulations

- If the rigid body is propagated in the tangent direction, numerical simulations showed the fact that the rigid body will become larger and larger.
- **Solution:** propagate in the secant direction.
Numerical Simulations

propagate by linear velocity
Numerical Simulations

propagate by linear velocity
Numerical Simulations

- Use the relationship between the euler angles and the angular velocity to update the euler angles in each time step.
- This keeps the shape of the rigid body. However, the matrix has a big probability to be singular when $\theta = n\pi$.
- Example: if we consider the initial position is where the three euler angles are all zero, then $\theta = 0$ and this leads to a singular matrix.
- **Solution:** no solution.
Numerical Simulations

- Use the relationship between the euler parameters and the angular velocity to update the euler parameters in each step.
- This keeps the shape of the rigid body. Also, it is shown in numerical simulation that the matrix can never be singular.
- Based on the initial euler angles $\phi = \theta = \varphi = 0$ and the relationship between euler parameters and euler angles, the initial values of the four euler parameters are $e_0 = 1, e_1 = e_2 = e_3 = 0$. 

propagate by euler parameters
Numerical Simulations

propagate by euler parameters
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propagate by euler parameters
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propagate by euler parameters
Numerical Simulations
propagate by euler parameters
Future Work

- different shapes of rigid body
- outside force to the rigid body
- fluid field in the simulation
- parachutists motion in the parachute inflation
References

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