1 The Exact DEs and Almost Exact DEs

General Formula:

\[ M(x, y)dx + N(x, y)dy = 0 \]

1.1 Exact

If \( M_y = N_x \), then the DE is called exact. Furthermore, the general solution is given by

\[ F(x, y) = C. \]

Do the total differentiation, we have

\[ F_x dx + F_y dy = 0. \]

1.1.1 Steps for Solving

- **SETP1**: Compare this equation with the general formula. If we want two match, then

  \[ F_x = M(x, y), \]
  \[ F_y = N(x, y). \]

- **SETP2**: Look at the equation (1), it can be solved by direct integration.

  \[ \int F_x dx = \int M(x, y)dx \]

  \[ F(x, y) = \int M(x, y)dx + g(y) \]

  **Note**: Or, you can look at the equation (2), it can also be solved by direct integration.

  \[ \int F_y dy = \int N(x, y)dy \]

  \[ F(x, y) = \int N(x, y)dx + g(x) \]

- **STEP3**: Then, plug (3) into (2) to obtain \( g(y) \).
  **Note**: Or, plug (4) into (1) to obtain \( g(x) \).
• SETP4: Finally, the solution is
\[ \int M(x,y)dx + g(y) = C. \]

**Note:** Or the solution is
\[ \int N(x,y)dy + g(x) = C. \]

### 1.1.2 Example 1

**Question:** Solving the following DE using the exact DE method

\([1 + \ln (xy)]dx + \frac{x}{y}dy = 0\)

**Solution:** First of all, we need to identify what is \(M(x,y)\) and \(N(x,y)\).

\[ M(x,y) = 1 + \ln(xy) \]
\[ N(x,y) = \frac{x}{y} \]

Then calculate \(M_y\) and \(N_x\).

\[ M_y = \frac{1}{y} = N_x \]

This tells that the DE is exact. Then we can follow the above steps.

• **STEP1:**

\[ F_x = M(x,y) = 1 + \ln(xy), \quad F_y = N(x,y) = \frac{x}{y} \]

• **STEP2:**

\[ F(x,y) = \int M(x,y)dx + g(y) \]
\[ = \int 1 + \ln(xy)dx + g(y) \]
\[ = x \ln(xy) + g(y) \]

• **STEP3:**

\[ F_y = \frac{x}{y} + g'(y) = \frac{x}{y} \]
\[ \Rightarrow g'(y) \Rightarrow g(y) = C \]

• **STEP4:** The solution is
\[ x \ln(xy) = C. \]
1.2 Not Exact

Then, can we still the DE if it is not exact? For two special cases, it can still be solved by integrating factor.

- 1. If \( \frac{M_y - N_x}{N} = f(x) \), then the integrating factor is \( \rho(x) = e^{\int f(x) dx} \);

- 2. If \( \frac{N_x - M_y}{M} = g(y) \), then the integrating factor is \( \rho(y) = e^{\int g(y) dy} \).

Then, the original DE, which is not exact, can be converted to an exact DE by multiplying the integrating factor \( \rho \).

1.2.1 Example 2

Question: Solving the following DE using the exact DE method

\[
(2x - y^2)dx + xydy = 0
\]

Solution: First, let us check if the DE is exact. We have

\[
M(x, y) = 2x - y^2, \quad N(x, y) = xy,
\]

\[
M_y = -2y; \quad N_x = y.
\]

Obviously, \( M_y \neq N_x \), the DE is not exact. However, we should notice the following.

\[
\frac{M_y - N_x}{N} = -3
\]

This is a single-variable function of \( x \). Let \( \rho(x) = e^{\int \frac{-3}{x} dx} = x^{-3} \) and then multiply it to the original DE.

\[
\frac{2x - y^2}{x^3} dx + \frac{y}{x^2} dy = 0
\]

We know the above DE is exact. Then we can follow the steps for exact DEs.

\[
F(x, y) = \int N(x, y) dy = \int \frac{y}{x^2} dy = \frac{y^2}{2x^2} + g(x)
\]

\[
\Rightarrow M(x, y) = F_x = -\frac{y^2}{x^3} + g'(x) = \frac{2x - y^2}{x^3}
\]

\[
\Rightarrow g'(x) = \frac{2}{x^2}
\]

\[
\Rightarrow g(x) = -\frac{2}{x} + C
\]

\[
\Rightarrow F(x, y) = \frac{y^2}{2x^2} - \frac{2}{x}
\]

Therefore, the solution is

\[
\frac{y^2}{2x^2} - \frac{2}{x} = C.
\]
2 General Formulas in Chapter 1

2.1 Direct Integral

\[ y' = f(x) \]
\[ \Rightarrow y(x) = \int f(x)dx + C \]

2.2 Separation of Variables

\[ y' = \frac{f(x)}{g(y)} \]
\[ \Rightarrow \int g(y)dy = \int f(x)dx \]

2.3 Integrating Factors

\[ y' + P(x)y = Q(x) \]
\[ \Rightarrow \frac{d}{dx} (\rho(x)y) = Q(x)\rho(x) \]
\[ \Rightarrow \rho(x)y = \int Q(x)\rho(x)dx + C \]

where the integrating factor is defined as

\[ \rho(x) = e^{\int P(x)dx}. \]

2.4 Polynomial Substitution

\[ y' = F(ax + by + c) \]

Introduce a new variable \( v = ax + by + c \).

2.5 \( \frac{y}{x} \) Substitution

\[ y' = F\left(\frac{y}{x}\right) \]

Introduce a new variable \( v = \frac{y}{x} \).

2.6 Bernoulli DEs

\[ y' + P(x)y = Q(x)y^n \]

Introduce a new variable \( v = y^{1-n} \).