1 Direct Integral

DEs with the following formula

\[ y^{(n)} = f(x) \]

can be solved by direct integral.
For example, if the order is 1 \((n = 1)\), then

\[
\begin{align*}
  y' &= f(x) \\
  \int y' \, dx &= \int f(x) \, dx \\
  \int \frac{dy}{dx} \, dx &= \int f(x) \, dx \\
  \int dy &= \int f(x) \, dx \\
  y &= F(x) + C
\end{align*}
\]

For this kind of DE with an order \(n\), you need to apply directly integral \(n\) times.

1.1 Problem 1.2.7

Question: find a function \(y = f(x)\) satisfying the given differential equation and the prescribed initial condition.

\[
\frac{dy}{dx} = \frac{10}{x^2 + 1}; \quad y(0) = 0
\]

Solution: According to the direct integral, the general solution is

\[
y(x) = \int f(x) \, dx + C = \int \frac{10}{x^2 + 1} + C, \\
= 10\arctan(x) + C.
\]

Use the initial condition \(y(0) = C = 0\). Therefore, the particular solution is \(y(x) = 10\arctan(x)\).
2 Separation of Variables

First-Order initial value problem (IVP):

\[
\begin{cases}
    y' = f(x, y) \\
    y(x = a) = b
\end{cases}
\]

If \( f(x, y) \) can be written as the following

\[
y' = \frac{g(x)}{f(y)}
\]

then the equation is said to be separable.

Solution:

\[
\int f(y) dy = \int g(x) dx \\
F(y) = G(x) + C
\]

where \( C \) is a constant determined by the initial condition: \( y(a) = b \).

2.1 Problem 1.4.1

Question: Find general solutions of the differential equations.

\[
\frac{dy}{dx} + 2xy = 0
\]

Solution:

\[
\text{SeparableDE} \Rightarrow \int \frac{dy}{y} = \int -2x dx \\
In(y) = -x^2 + In(C) \\
y = Ce^{-x^2}
\]

which is the general solution of the DE (explicitly).

2.2 Problem 1.4.13

Question: Find general solutions of the differential equations.

\[
y^3 \frac{dy}{dx} = (y^4 + 1) \cos(x)
\]
Solution:

\[ y^3 \frac{dy}{dx} = (y^4 + 1) \cos(x) \]

\[
\frac{\text{SeparableDE}}{\int} \frac{y^3 dy}{y^4 + 1} = \int \cos(x) \, dx \\
\frac{1}{4} \ln(y^4 + 1) = \sin(x) + C_1 \\
y^4 + 1 = Ce^{\sin(x)}
\]

which is the general solution of the DE (implicitly).

2.3 Problem 1.4.19

Question: Find particular solutions of the differential equations.

\[ \frac{dy}{dx} = ye^x, \; y(0) = 2e \]

Solution:

\[ \frac{dy}{dx} = ye^x \frac{\text{SeparableDE}}{\int} \frac{dy}{y} = \int e^x \, dx \\
\ln(y) = e^x + C_1 \\
y = Ce^e
\]

which is the general solution of the DE (explicitly). Then, use the initial condition to determine the value of the constant.

\[ y(0) = Ce^{0} = Ce = 2e \Rightarrow C = 2 \]

\[ \Rightarrow y(x) = 2e^{e^x} \]

3 Natural Growth and Decay

The differential equation

\[ \frac{dx}{dt} = kx \; (k \; a \; constant) \]

is the mathematical model for natural growth and decay problems. The general solution is given by

\[ \frac{dx}{x} = k \, dt \Rightarrow x(t) = x_0e^{kt} \]

where \( x_0 \) is the initial population. When \( k > 0 \), it is called natural growth; when \( k < 0 \), it is called natural decay.
3.1 Problem 1.4.34

Question: (Population Growth) In a certain culture of bacteria, the number of the bacteria increased sixfold in 10 h. How long did it take for the population to double?

Solution: The differential equation and the general solution is

\[
\frac{dx}{dt} = kx \Rightarrow x(t) = x_0 e^{kt}.
\]

"The number of the bacteria increased sixfold in 10 h" tells that \(x(10) = 6x_0\).

\[
x(10) = x_0 e^{10k} = 6x_0 \Rightarrow k = \frac{\ln(6)}{10}
\]

Then, solve the algebraic equation \(x(t) = x_0 e^{kt} = 2x_0\) for the time \(t\).

\[
x_0 e^{kt} = 2x_0 \Rightarrow e^{kt} = 2
\]

\[
\Rightarrow t = \frac{\ln(2)}{k} = \frac{10 \times \ln(2)}{\ln(6)} \approx 3.87 \text{ h}
\]