1 Basic of DE Systems

- Express a higher order linear DE as a system of first order DEs.
- Express a system of second order DEs as a system of first order DEs.

1.1 Extra Example

Question: Convert the following DE as a system of first order DEs

\[ x''' + 3x'' + 3x' + x = t^2 + 2 \]

Solution: Let \( y_1 = x, y_2 = x', y_3 = x'' \). We then get the system

\[
\begin{align*}
    y_1' &= y_2 \\
    y_2' &= y_3 \\
    y_3' &= y_1 + 3y_3 + 3y_2 + y_1 = t^2 + 2
\end{align*}
\]

1.2 Extra Example

Question: Convert the following system of second order DEs as a system of first order DEs

\[
\begin{align*}
    x + 3y' + 5x'' &= f(t) \\
    y + 2x' + 4y'' &= g(t)
\end{align*}
\]

Solution: Let \( u_1 = x, u_2 = x', u_3 = y, u_4 = y' \). We then get the system

\[
\begin{align*}
    u_1' &= u_2 \\
    u_2' &= u_4 \\
    u_3' &= 3u_4 + 5u_2' = f(t) \\
    u_4' &= 2u_2 + 4u_4' = g(t)
\end{align*}
\]

2 Spring-Mass System

Use Newton’s Second Law \( ma = F \) to derive the system of DEs.

2.1 Problem 4.1.24

Question: Write the system of ODEs for the displacement \( x_1(t) \) and \( x_2(t) \) for the spring-mass system shown below.
Solution: According to Newton’s Second Law, \( mx'' = ma = F \).

- We move block 1 by \( x_1 \). So the spring 1 is extended by \( x_1 \).

- We move block 2 by \( x_2 \), So the spring 2 is extended by \( x_2 - x_1 \).

- The spring 3 is extended by \( -x_2 \).

We get the system

\[
\begin{aligned}
    m_1 x_1'' &= -k_1 x_1 + k_2 (x_2 - x_1) \\
    m_2 x_2'' &= -k_2 (x_2 - x_1) + k_3 (-x_2)
\end{aligned}
\]

3 Substitution/Elimination Method

Solution Steps

- Step 1: Express one dependent variable by the other by solving the two DEs, i.e.,
decoupling variables.

- Step 2: Substitute the above variable to one DE to obtain a single-dependent variable
DE.

- Step 3: Solve the resulting sing-dependent variable DE from the above Step 2.

- Step 4: back substitute to solve for the other dependent variables

3.1 Problem 4.1.18

Question: Find the particular solution of the DEs system

\[
\begin{aligned}
    x' &= -y \\
    y' &= 10x - 7y
\end{aligned}
\]

Solution: Follow the above steps. Step 1: From the first DE, we have \( y = -x' \) and also \( y' = -x'' \).

Step 2: Substitute in the the second DE.

\[
\begin{aligned}
    y' &= 10x - 7y \\
    \Rightarrow -x'' &= 10x - 7(-x')
\end{aligned}
\]
\[ x'' + 7x' + 10x = 0 \]

Step 3: Solve the single-dependent variable DE in Step 2.

\[ r^2 + 7r + 10 = 0 \]
\[ \Rightarrow r_1 = -2, r_2 = -5 \]
\[ \Rightarrow x = c_1 e^{-2t} + c_2 e^{-5t} \]

Step 4: Substitute \( x_2 \) to the relation in Step 1.

\[ y = -x' \]
\[ \Rightarrow y = 2c_1 e^{-2x} + 5c_2 e^{-5t} \]