1 Second Order, Constant Coefficient DEs

General formula

\[ ay'' + by' + cy = 0 \]

Characteristic equation

\[ ar^2 + br + c = 0 \]

There are three cases for the roots of the characteristic equation.

- 1. \( b^2 - 4ac > 0 \) There are two distinct roots \( r_1 \) and \( r_2 \). The general solution is given by

\[ y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}. \]

- 2. \( b^2 - 4ac = 0 \) There are two equal roots \( r_1 = r_2 = r \). The general solution is given by

\[ y(x) = (C_1 + C_2 x) e^{rx}. \]

- 3. \( b^2 - 4ac < 0 \) There are two complex roots \( r_1 = \alpha + i\beta \) and \( r_2 = \alpha - i\beta \). The general solution is given by

\[ y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)). \]

2 Wronskian

\[
W(y_1, \ldots, y_n) = \begin{vmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}_{n \times n}
\]

For \( n \) functions \( y_1, y_2, \ldots, y_n \), if the Wronskian \( W(y_1, \cdots, y_n) \neq 0 \), then they are linearly independent. Conversely, if \( y_1, y_2, \cdots, y_n \) are linearly dependent, then the Wronskian \( W(y_1, \cdots, y_n) = 0 \).
3 Solution of Non-homogeneous Linear Higher Order DEs

Theorem 1. If $y_c$ is a general solution to $Ly = 0$ and $y_p$ is a particular solution to $Ly = f(x)$, then $y = y_c + y_p$ is the general solution to the non-homogeneous DE $Ly = f(x)$.

3.1 Undetermined Coefficients Method

Objective: Solve the non-homogeneous DE $Ly = f(x)$.

- Step 1: Use the characteristic equation to solve the homogeneous DE $Ly = 0$, and find the linearly independent solutions $y_1, y_2, \cdots$.
- Step 2: Guess the formula of the particular solution $y_p$ according to the formula of $f(x)$. Pay attention to the cases of duplication where the initial guess is multiplied by $x^s$. There are undetermined coefficients in $y_p$.
- Step 3: Substitute the particular solution $y_p$ into the non-homogeneous DE $Ly = f(x)$ and determine the coefficients.
- Step 4: Theorem 1 tells that the general solution to the non-homogeneous DE $Ly = f(x)$ is given by $y = y_c + y_p$.

The following table lists the functions that can be chosen as the particular solutions for a given RHS. Three fundamental cases are listed.

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$y_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_0 + b_1 x + \cdots + b_m x^m$</td>
<td>$x^s[A_0 + A_1 x + \cdots + A_m x^m]$</td>
</tr>
<tr>
<td>$\cos(kx) + b\sin(kx)$</td>
<td>$x^s[\cos(kx) + B\sin(kx)]$</td>
</tr>
<tr>
<td>$ae^{rx}$</td>
<td>$x^s[ae^{rx}]$</td>
</tr>
<tr>
<td>$(\cos(kx) + \sin(kx))(c_0 + \cdots + c_m x^m)$</td>
<td>$x^s(\cos(kx) + B\sin(kx))[C_0 + \cdots + C_m x^m]$</td>
</tr>
<tr>
<td>$ae^{rx}(b_0 + b_1 x + \cdots + b_m x^m)$</td>
<td>$x^s e^{rx}[A_0 + A_1 x + \cdots + A_m x^m]$</td>
</tr>
<tr>
<td>$ae^{rx}(\cos(kx) + \sin(kx))$</td>
<td>$x^s e^{rx}[\cos(kx) + B\sin(kx)]$</td>
</tr>
</tbody>
</table>

How to determine the value of duplication $s$? First, we initially guess the formula of $y_p$ according to $f(x)$ without considering any duplication. Then, compare the formula of $y_p$ you initially guess with $y_c$. If there is some part that appears or partially appears in $y_c$, then multiply a $x$ to that part. Repeat this until there is no common term between $y_p$ and $y_c$.

4 Variation of Parameters

Consider a 2nd order non-homogeneous DE

$$y'' + P(x)y' + Q(x)y = f(x)$$

Compose a formula for the particular solution for the above DE.

- Step 1: Find the general solution for the homogeneous DE: $y_c = C_1 y_1 + C_2 y_2$.
- Step 2: Assume the particular solution is $y_p = u_1(x)y_1 + u_2(x)y_2$.

Even if only $\cos(kx)$ function or only $\sin(kx)$ function appears in $f(x)$, we still need to include both of them in the particular solution $y_p$. 

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1 Even if only $\cos(kx)$ function or only $\sin(kx)$ function appears in $f(x)$, we still need to include both of them in the particular solution $y_p$. 

• Step 3: Obtain a linear system

\[
\begin{cases}
y_1' u_1(x) + y_2' u_2(x) = 0 \\
y_1' u_1(x) + y_2' u_2(x) = f(x)
\end{cases}
\]

which can also be written as

\[
\begin{pmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{pmatrix}
\begin{pmatrix}
u_1'(x) \\
u_2'(x)
\end{pmatrix}
= \begin{pmatrix}0 \\ f(x)\end{pmatrix}
\]

• Step 3: Calculate Wronskian \(W(y_1, y_2)\)

• Step 4: Solve the linear system

\[
\begin{align*}
u_1(x) &= \int -\frac{y_2 f(x)}{W(y_1, y_2)} \, dx \\
u_2(x) &= \int \frac{y_1 f(x)}{W(y_1, y_2)} \, dx
\end{align*}
\]

• Step 5: \(y_p = u_1(x)y_1 + u_2(x)y_2\)

5 Initial Value Problems

• step 1. Solve the Homogeneous DE by characteristic equation and obtain \(y_c(x)\).

• step 2. Guess and solve for the particular solution \(y_p(x)\).

• step 3. Write down the general solution \(y(x) = y_c(x) + y_p(x)\).

• step 4. Use initial conditions to determine the constants in \(y(x)\).

6 Euler Equations

Do substitution \(y(x) = x^r\)