AMS 361 Midterm Exam 2 Solutions

April 17, 2014

1)

\[ y'' + 5y' + 4y = 0 \]

\[ y(0) = 2 \]

\[ y'(0) = 2 \]

Characteristic equation is

\[ r^2 + 5r + 4 = 0 \]

So \( r = -4, -1. \)

\[ y(x) = c_1 e^{-4x} + c_2 e^{-x} \]

Now apply the initial conditions:

\[ y(0) = c_1 + c_2 = 2 \]

\[ y'(0) = -4c_1 - c_2 = 2 \]

Which implies \( c_1 = -\frac{4}{3}, c_2 = \frac{10}{3}. \)

\[ y(x) = -\frac{4}{3} e^{-4x} + \frac{10}{3} e^{-x} \]

2)

\[ y^{(4)} - 2y'' + y = 0 \]

Characteristic equation is

\[ r^4 - 2r^2 + 1 = 0 \]

\[ (r^2 - 1)^2 = 0 \]

So \( r = \pm 1 \) each with multiplicity 2.
These functions are linearly dependent. To see this by definition, recall

\[
\cos 2x = 1 - 2\sin^2 x
\]

So that

\[
y_1(x) - \frac{1}{2}y_3(x) + 2y_2(x) = 0
\]

4)

\[
x^2 y'' - 2xy' - 10y = 0
\]

Substitute \( y = x^r \) and we want to find \( r \).

\[
x^2 r (r - 1)x^{r-2} - 2x^r x^{r-1} - 10x^r = 0
\]

\[
r(r - 1) - 2r - 10 = 0
\]

\[
r^2 - 3r - 10 = 0
\]

\[
(r - 5)(r + 2) = 0
\]

So \( r = 5, -2 \) and

\[
y(x) = c_1 x^5 + c_2 x^{-2}
\]

5)

\[
y'' - 3y' + 2y = 2x + \cos 2x
\]

To get \( y_c \):

\[
r^2 - 3r + 2 = 0
\]

\[
(r - 2)(r - 1) = 0
\]
So $r = 2, 1$ and the complementary solution is

$$y_c = c_1 e^{2x} + c_2 e^x$$

Now guess

$$y_p = Ax + B + C \cos 2x + D \sin 2x$$

Plugging this in results in

$$-3A + 2B + 2Ax - 2(C + 3D) \cos 2x + (6C - 2D) \sin 2x = 2x + \cos 2x$$

Equating coefficients results in

$$A = 1$$

$$B = \frac{3}{2}$$

$$C = -\frac{1}{20}$$

$$D = -\frac{3}{20}$$

So

$$y_p = x + \frac{3}{2} - \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x$$

6) 

$$y''' - y' = e^x$$

$$y(0) = y'(0) = 0$$

$$y''(0) = 1$$

To get $y_c$:

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0$$
\[ r(r - 1)(r + 1) = 0 \]

\[ y_e = c_1 + c_2 e^x + c_3 e^{-x} \]

Now guess (due to the duplication in \( y_e \)):

\[ y_p = B x e^x \]

Plugging this in gives

\[ 2B e^x = e^x \]

So

\[ B = \frac{1}{2} \]

\[ y_p = \frac{1}{2} x e^x \]

The general solution is

\[ y(x) = c_1 + c_2 e^x + c_3 e^{-x} + \frac{1}{2} x e^x \]

The initial conditions are

\[ y(0) = c_1 + c_2 + c_3 = 0 \]

\[ y'(0) = c_2 - c_3 + \frac{1}{2} = 0 \]

\[ y''(0) = c_2 + c_3 + 1 = 1 \]

Which gives \( c_1 = 0 \), \( c_2 = -\frac{1}{4} \), \( c_3 = \frac{1}{4} \).

\[ y(x) = -\frac{1}{4} e^x + \frac{1}{4} e^{-x} + \frac{1}{2} x e^x \]
\[ y'' + y = \tan x \]

The right hand side is not easily handled with undermined coefficients. Variation of parameters is more appropriate.

\[ y_c = c_1 \sin x + c_2 \cos x \]

The Wronskian is

\[ W(x) = \det \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} = -\sin^2 x - \cos^2 x = -1 \]

Applying variation of parameters,

\[ y_p(x) = -\sin x \int \frac{\cos(x)\tan(x)}{-1} \, dx + \cos x \int \frac{\sin(x)\tan(x)}{-1} \, dx \]

\[ = \sin x \int \sin x \, dx - \cos x \int \frac{\sin^2 x}{\cos x} \, dx \]

\[ = -\sin(x)\cos(x) - \cos(x) \int |\sec(x) - \cos(x)| \, dx \]

\[ = -\sin(x)\cos(x) - \cos(x)[\ln|\sec x + \tan x| - \sin x] \]

\[ = -\cos(x)\ln|\sec(x) + \tan(x)| \]