1 Second Order, Constant Coefficient DEs

General formula

\[ ay'' + by' + cy = 0 \]

Characteristic equation

\[ ar^2 + br + c = 0 \]

There are three cases for the roots of the characteristic equation.

- 1. \( b^2 - 4ac > 0 \) There are two distinct roots \( r_1 \) and \( r_2 \). The general solution is given by

  \[ y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}. \]

- 2. \( b^2 - 4ac = 0 \) There are two equal roots \( r_1 = r_2 = r \). The general solution is given by

  \[ y(x) = (C_1 + C_2 x) e^{rx}. \]

- 3. \( b^2 - 4ac < 0 \) There are two complex roots \( r_1 = \alpha + i\beta \) and \( r_2 = \alpha - i\beta \). The general solution is given by

  \[ y(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)). \]

1.1 Problem 3.3.12

Question: Find the general solution of the following DE

\[ y'' - 2y' + 2y = 0 \]

Solution: The characteristic equation is

\[ r^2 - 2r + 2 = 0 \]

\[ \Rightarrow r_1 = 1 + i, r_2 = 1 - i \]

\[ \Rightarrow y = e^{x}(C_1 \cos(x) + C_2 \sin(x)) \]
1.2 Problem 3.3.19

Question: Find the general solution of the following DE

\[ 35y'' - y' - 12y = 0 \]

Solution: The characteristic equation is

\[ 35r^2 - r - 12 = 0 \]

\( \Rightarrow r_1 = \frac{3}{5}, r_2 = -\frac{4}{7} \)

\( \Rightarrow y = C_1 e^{\frac{3}{5}x} + C_2 e^{-\frac{4}{7}x} \)

2 Higher Order, Constant Coefficient DEs

The general idea is similar to the second order DE.

- step 1: write down the characteristic equation.
- step 2: solve the characteristic equation for \( r \).
- step 3: find the general solution according to the three different cases.

2.1 Problem 3.3.3

Question: Find the general solution of the following DE

\[ 3y''' + 2y'' = 0 \]

Solution: The characteristic equation is

\[ 3r^3 + 2r^2 = 0 \]

\( \Rightarrow r_1 = r_2 = 0, r_3 = -\frac{2}{3} \)

\( \Rightarrow y = (C_1 + C_2x)e^{0x} + C_3 e^{-\frac{2}{3}x} \)

2.2 Problem 3.3.14

Question: Find the general solution of the following DE

\[ (D - 1)(D - 2)(D - 3)(D + 1)y = 0 \]

Solution: The characteristic equation is

\[ (r - 1)(r - 2)(r - 3)(r + 1) = 0 \]

\( \Rightarrow r_1 = 1, r_2 = 2, r_3 = 3, r_4 = -1 \)

\( \Rightarrow y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x} + C_4 e^{-x} \)
2.3 Problem 3.3.6

Question: Find the general solution of the following DE

\[ y^{(4)} = y''' + y'' + y' + 2y \]

Solution: The characteristic equation is

\[ r^4 = r^3 + r^2 + r + 2 \]

\[ \Rightarrow (r + 1)(r - 2)(r^2 + 1) = 0 \]

\[ \Rightarrow r_1 = -1, r_2 = 2, r_3 = i, r_4 = -i \]

\[ \Rightarrow y = C_1e^{-x} + C_2e^{2x} + e^{0x}(C_3cos(x) + C_4sin(x)) \]

2.4 Extra Problem 1

Question: Find the general solution of the following DE

\[ (D - 1)^3(D + 1)y = 0 \]

Solution: The characteristic equation is

\[ (r - 1)^3(r + 1) = 0 \]

\[ \Rightarrow r_1 = r_2 = r_3 = 1, r_4 = -1 \]

\[ \Rightarrow y = (C_1 + C_2x + C_3x^2)e^x + C_4e^{-x} \]

2.5 Extra Problem 2

Question: Find the general solution of the following DE

\[ y^4 + 8y'' + 16y = 0 \]

Solution: The characteristic equation is

\[ r^4 + 8r^2 + 16 = 0 \]

\[ \Rightarrow (r^2 + 4)^2 = 0 \]

\[ \Rightarrow r_1 = r_2 = 2i, r_3 = r_4 = -2i \]

\[ \Rightarrow y = (C_1 + C_2x)cos(2x) + (C_3 + C_4x)sin(2x) \]