1 Population Model

- $\beta(t)$ is the number of births per unit population per unit time at time $t$;
- $\delta(t)$ is the number of deaths per unit population per unit time at time $t$.

1.1 General Population Equation

$$\frac{dP}{dt} = (\beta - \delta)P,$$

where $\beta = \beta(t)$ and $\delta = \delta(t)$ can be either a constant or functions of $t$ or they may, indirectly, depend on the unknown function $P(t)$.

1.2 The Logistic Equation

In this case, the birth rate $\beta(t)$ is a linear decreasing function of the population size $P$ so that $\beta = \beta_0 - \beta_1P$ where $\beta_0$ and $\beta_1$ are positive constants. As for the death rate, it remains a constant.

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta)P$$

The differential equation can be rewritten as

$$\frac{dP}{dt} = kP(M - P),$$

where $M$ is called the carrying capacity of the environment, that is to say, the maximum population that the environment can support on a long-term basis.

1.3 Doomsday vs. Extinction

Here, we assume the birth occur at the rate of $kP^2$ (per unit time and $k$ is constant). The birth rate (births per unit time per population) is then given by $\beta = kP$. And the death rate remains a constant.

$$\frac{dP}{dt} = kP^2 - \delta P$$

The differential equation can be rewritten as

$$\frac{dP}{dt} = kP(P - M),$$

where $M$ is called threshold population.
1.4 Problem 2.1.1

Question: Suppose that when a certain lake is stoked with fish, the birth and death rates $\beta$ and $\delta$ are both inversely proportional to $\sqrt{P}$.

- (a) show that $P(t) = (\frac{1}{2}kt + \sqrt{P_0})^2$, where $k$ is constant.
- (b) If $P_0 = 100$ and after 6 months there are 169 fish in the lake, how many will there be after 1 year?

Solution: (a) Since the birth and death rates $\beta$ and $\delta$ are both proportional to $\sqrt{P}$, we have

$$\beta = \frac{k_1}{\sqrt{P}}$$
$$\delta = \frac{k_2}{\sqrt{P}}$$

Plug into the general population model

$$\frac{dP}{dt} = (\beta - \delta)P$$

$$\Rightarrow \frac{dP}{dt} = (k_1 - k_2)\sqrt{P}$$

$$\Rightarrow \frac{dP}{dt} = k\sqrt{P}$$

where $k = k_1 - k_2$ is constant.

Then this differential equation can be solved by separation of variables.

$$P(t) = (\frac{1}{2}kt + C)^2$$

Plugging $P(0) = P_0$ into it, we can get $C^2 = P_0$. Thus

$$P(t) = (\frac{1}{2}kt + \sqrt{P_0})^2$$

(b) Given $P_0 = 100$ and $P(t = 6) = 169$, we can get

$$169 = (\frac{1}{2} \times 6k + \sqrt{100})^2.$$  

This gives $k = 1$. Thus, $P(12) = 256$.

2 Acceleration-Velocity Model

2.1 Velocity and Acceleration Models

$$v = \frac{dy}{dt}$$
$$a = \frac{dv}{dt}$$
where $a(t)$ is acceleration, $v(t)$ is velocity and $y(t)$ is position.

Newton’s Second Law of Motion gives

$$F = ma,$$
$$\Rightarrow m\frac{dv}{dt} = F,$$

where $F$ is the force.

### 2.2 Air Resistance Model

$$m\frac{dv}{dt} = F_G + F_R$$

where $F_R = kv^p$ with $1 \leq p \leq 2$, and the value of $k$ depends on the size and shape of the body as well as the density and velocity of the air.

In the case $p = 1$, the differential equation can be rewritten as

$$m\frac{dv}{dt} = -kv - mg$$
$$\Rightarrow \frac{dv}{dt} = -\frac{k}{m}v - g$$

Then, the DE is separable and the solution is given by

$$v(t) = (v_0 + \frac{g}{r})e^{-\rho t} - \frac{g}{\rho}$$

where $v(0) = v_0$ is the initial velocity of the body.

The terminal speed of a falling body with air resistance is defined as

$$v_r = \lim_{t \to \infty} v(t) = -\frac{g}{\rho}$$

Then the velocity can be rewritten as

$$v(t) = (v_0 - v_r)e^{-\rho t} + v_r.$$  

According to the relation between velocity and position, the position of the body is given by integrating the velocity.

$$\frac{dy}{dt} = v$$
$$\Rightarrow y = \int (v_0 - v_r)e^{-\rho t} + v_r dt$$
$$\Rightarrow y = -\frac{1}{\rho}(v_0 - v)r e^{-\rho t} + v_r t + C$$

With the initial height of the body given by $y(0) = y_0$, we know

$$y(t) = y_0 + v_r t + \frac{q}{\rho}(v_0 - v_r)(1 - e^{-\rho t}).$$
2.3 Gravitational Acceleration

In this model, the force is given by the gravitational force of attraction between two point masses $M$ and $m$ located at a distance $r$ apart

$$F = \frac{GMm}{r^2}$$

2.4 Problem 2.2.1

Question: Suppose that a body moves horizontally through a medium whose resistance is proportional to its velocity $v$, so that $v' = -kv$.

- (a) Show that its velocity and position at time $t$ are given by
  $$v(t) = v_0 e^{-kt}$$
  $$x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt})$$

- (b) Conclude that the body travels only a finite distance, and find that distance.

Solution: (a) Solve the differential equation with initial condition $v(0) = v_0$ by separation of variables. It gives

$$v(t) = v_0 e^{-kt}.$$ 

Then, integrate the velocity with initial condition $x(0) = x_0$ to obtain the position of the body and we have

$$x(t) = x_0 + \left(\frac{v_0}{k}\right)(1 - e^{-kt}).$$

(b) As $t \rightarrow \infty$, we know that $e^{-kt} \rightarrow 0$. Thus,

$$x \rightarrow x_0 + \frac{v_0}{k}, \quad t \rightarrow \infty.$$ 

3 Application to Finance

We consider the following terms:

- $z(t)$ is the amount owed to the bank at time $t$
- $r$ is the interest rate which is a constant
- $dt$ is the time period from the time interval $[t, t + dt]$
- $p$ is the payment rate which is also a constant
- $dz$ is the change of debts to bank during time $dt$.

Then the decrement of loan due to payments made to the bank is given by

$$dz = z(t)rdt - wdt$$

$$\Rightarrow z' = zr - w$$
4 Hyperbolic Trigonometric Functions

The hyperbolic functions are

- Hyperbolic sine:
  \[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]

- Hyperbolic cosine:
  \[ \cosh(x) = \frac{e^x + e^{-x}}{2} \]

- Hyperbolic tangent:
  \[ \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]

- Hyperbolic cotangent:
  \[ \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \]

- Hyperbolic secant:
  \[ \sech(x) = (\cosh(x))^{-1} = \frac{2}{e^x + e^{-x}} \]

- Hyperbolic cosecant:
  \[ \csch(x) = (\sinh(x))^{-1} = \frac{2}{e^x - e^{-x}} \]