1 Basic of DE Systems

- Express a higher order linear DE as a system of first order DEs.

- Express a system of second order DEs as a system of first order DEs.

1.1 Extra Example

Question: Convert the following DE as a system of first order DEs

\[ x'''' + 3x'' + 3x' + x = t^2 + 2 \]

Solution: Let \( y_1 = x, y_2 = x', y_3 = x'' \). We then get the system

\[
\begin{cases}
  y_1' = y_2 \\
  y_2' = y_3 \\
  y_3' + 3y_3 + 3y_2 + y_1 = t^2 + 2
\end{cases}
\]

1.2 Extra Example

Question: Convert the following system of second order DEs as a system of first order DEs

\[
\begin{cases}
  x + 3y' + 5x'' = f(t) \\
  y + 2x' + 4y'' = g(t)
\end{cases}
\]

Solution: Let \( u_1 = x, u_2 = x', u_3 = y, u_4 = y' \). We then get the system

\[
\begin{cases}
  u_1' = u_2 \\
  u_2' = u_4 \\
  u_1 + 3u_4 + 5u_2' = f(t) \\
  u_3 + 2u_2 + 4u_4' = g(t)
\end{cases}
\]

2 Spring-Mass System

Use Newton’s Second Law \( ma = F \) to derive the system of DEs.
2.1 Problem 4.2.1

Question: Write the system of ODEs for the displacement $x_1(t)$ and $x_2(t)$ for the spring-mass system shown below.

Solution: According to Newton’s Second Law, $m \dddot{x} = ma = F$.

- We move block 1 by $x_1$. So the spring 1 is extended by $x_1$.
- We move block 2 by $x_2$. So the spring 2 is extended by $x_2 - x_1$.
- The spring 3 is extended by $-x_2$.

We get the system

$$\begin{cases} m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \ddot{x}_2 = -k_2(x_2 - x_1) + k_3(-x_2) \end{cases}$$

3 Substitution/Elimination Method

Solution Steps

- Step 1: Express one dependent variable by the other by solving the two DEs, i.e., decoupling variables.
- Step 2: Substitute the above variable to one DE to obtain a single-dependent variable DE.
- Step 3: Solve the resulting sing-dependent variable DE from the above Step 2.
- Step 4: Back substitute to solve for the other dependent variables.
3.1 Problem 4.3.1

Question: Find the particular solution of the DEs system

\[
\begin{cases}
  x' = -y \\
y' = 10x - 7y \\
x(0) = 2 \\
y(0) = -7
\end{cases}
\]

Solution: Follow the above steps. Step 1: From the first DE, we have \( y = -x' \) and also \( y' = -x'' \).

Step 2: Substitute in the second DE.

\[
y' = 10x - 7y \\
\implies -x'' = 10x - 7(-x') \\
\implies x'' + 7x' + 10x = 0
\]

Step 3: Solve the single-dependent variable DE in Step 2.

\[
r^2 + 7r + 10 = 0 \\
\implies r_1 = -2, r_2 = -5 \\
\implies x = c_1 e^{-2t} + c_2 e^{-5t}
\]

Step 4: Substitute \( x_2 \) to the relation in Step 1.

\[
y = -x' \\
\implies y = 2c_1 e^{-2x} + 5c_2 e^{-5t}
\]

Step 5: Use initial conditions to solve for the constants.

\[
\begin{cases}
  x(0) = c_1 + c_2 = 2 \\
y(0) = 2c_1 + 5c_2 = -7
\end{cases}
\implies \begin{cases}
  c_1 = \frac{17}{3} \\
c_2 = -\frac{11}{3}
\end{cases}
\implies \begin{cases}
  x = \frac{17}{3} e^{-2t} - \frac{11}{3} e^{-5t} \\
y = \frac{34}{3} e^{-2t} - \frac{55}{3} e^{-5t}
\end{cases}
\]